2D HIGHER ORDER BOUSSINESQ EQUATIONS FOR WAVES IN FLOWS WITH VORTICITY

Z. L. Zou

State Key Laboratory of Coastal and Offshore Engineering

Dalian University of Technology, Dalian 116024, P R China. zlzou@dlut.edu.cn

J. T. Kirby F. Y. Shi

Center for Applied Coastal Research, University of Delaware

Newark, DE 19716, USA. kirby@udel.edu, fyshi@dune.coastal.udel.edu

Abstract: This paper considers the higher order Boussinesq equations for the flow with vorticity, such as the wave propagation in shear currents or in surf zone. In order to show the characteristics of the mathematical model, the equations are first derived for a simple case: vertical two-dimension, horizontal bottom and constant vorticity. Then, the form of the equations for horizontal 2D case is given. Comparison with the Veeramony and Svendsen's horizontal 1D model is made. The determination of the horizontal vorticity component is discussed in order to make the closure of the equations.

Introduction

For water wave motions in currents or surf zone, vorticity is present. The mathematical model designed for such a case should take the vorticity into account and the usual mathematical model is not valid due to the assumption of irrotational flows. This study derives the mathematical model for the wave motions in the flows with vorticity. This problem is concerned in many engineering problems, such as the interaction of waves with shear currents, and the wave motions in surf zone and related nearshore circulations (Peregrine, 1998,1999).

Most Boussinesq models are based on the potential theory or a very weak vorticity condition (in which only the vertical component of vorticity is allowed present but the horizontal not). The wave breaking is considered approximately. One way is to adopt the eddy viscosity concept, and an artificial eddy viscosity term is added to the momentum equations (Zelt 1991; Karambas and Koutitas 1992; Wei et al. 1995). Another way is to use the surface roller concept (Deigaad and Fredsoe 1989; Schaffer et al 1993; Madsen et al 1997), and the roller is assumed to be the volume of water riding on the front face of the wave and propagating at the wave speed. So the ununiform velocity distribution is formed for breaking waves, which consists of the a constant value in the roller region and another ordinary wave velocity distribution, usually also constant value, in the rest part of total water depth, hence an excess momentum flux is introduced, which simulates wave breaking.

The fluid motions in breaking waves are essential turbulent, which are full of vorticity. The surface roller model for breaking waves is too simple to consider the breaking wave motions in view of more realistic description. A more accurate way is to take the rotational flows caused by the turbulent motions into account for Boussinesq models. Yu and Svendsen (1995), Veeramony and Svendsen (2000) derive the breaking terms in the Boussinesq model directly by assuming that the flow field is rotational. The excess pressure and momentum terms are introduced. The effect of breaking waves can be taken into account. This model is only for vertical two dimensional problems. Rego, Kirby and Thompson (2001) also give this kind of model, but in term of the velocity at arbitrary level. Shen (2000) presents the derivation of Boussinesq model to encompass the irrotational and rotational flow. The derivation is accurate to $O(\mu^2)$ (μ is the ratio of water depth to wave length), the nonlinear parameter (the ratio of wave height to water depth) and the vorticity can have the order of magnitude of $O(1)$ accurate to $O(\mu^2)$. The governing equation is not given in an explicit form in this derivation.

The present study presents the higher order Boussinesq model in which vorticity is considered. First, we derive the model for the vertical two dimensional, constant vorticity and water depth case. From this simple case, we can get a quick understanding of the equations we want to set up. Then, the model is extended to the horizontal two dimensional and variable water depth case, For the model, the nonlinear parameter can have the order of magnitude of $O(1)$ accurate to $O(\mu^2)$, and the dispersion property is accurate to $O(\mu^4)$. In order to get the closure of the model, the determination of the vertical distribution of vorticity is discussed.

The derivation of the equations for a simple case

The derivation of equations

We first derive the equations for the following simple case: the vertical two dimensional flow on horizontal bottom and with constant vorticity Ω over water depth. The velocity produced by the vorticity has the expression

$$U(z) = \Omega(z+h) \tag{1.1}$$

Since the problem is vertically two-dimensional, the vorticity will keep constant if it is constant initially and always remains in the current flow afterward. So the wave motions will remain potential flows and have the potential function ϕ . The Bernoulli equation for this flow reads

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}(u^2+w^2)+\frac{p}{\rho}+gz-\Omega\Psi=\frac{1}{2}U(0)^2+\frac{p_a}{\rho} \tag{1.2}$$

where p_a is atmospheric pressure, Ψ is the stream function, u and w are the horizontal and vertical velocity components respectively. On the free surface, $p = p_a$, this equation becomes

$$\left(\frac{\partial\phi}{\partial t}\right)_{z=\eta}+\frac{1}{2}(u_\eta^2+w_\eta^2)+g\eta-\Omega\Psi_\eta=\frac{1}{2}U(0)^2 \tag{1.3}$$

where the subscribe η means the value at free surface and we have

$$u_\eta = U(\eta)+\left(\frac{\partial\phi}{\partial x}\right)_{z=\eta}, \qquad w_\eta=\left(\frac{\partial\phi}{\partial z}\right)_{z=\eta} \tag{1.4}$$

From the above definition, we can get the potential flow velocity $u_p=\partial\phi/\partial x$ and $w_p=\partial\phi/\partial z$, the rotational flow velocity $u_r=U(z)$ and $w_r=0$. Then, the depth-averaged velocity will have the expression

$$\bar{u}=\bar{u}_p+\bar{u}_r \tag{1.5}$$

where

$$\bar{u}=\frac{1}{d}\int_{-h}^{\eta}u\,\mathrm{d}z\,, \qquad \bar{u}_p=\frac{1}{d}\int_{-h}^{\eta}\frac{\partial\phi}{\partial x}\mathrm{d}z \tag{1.6a,b}$$

$$\bar{u}_r=\frac{1}{d}\int_{-h}^{\eta}U(z)\,\mathrm{d}z=\frac{1}{2}\Omega d \tag{1.6c}$$

For a horizontal bottom, the velocity potential has the following expansion of a power series in vertical coordinate z.

$$\phi(x,z,t)=\phi_0-\frac{1}{2!}(h+z)^2\phi_{0xx}+\frac{1}{4!}(h+z)^4\phi_{0xxxx}+\cdots \tag{1.7}$$

here $\phi_0=\phi(x,z,t)\big|_{z=0}$. Hence we have

$$\left(\frac{\partial\phi}{\partial x}\right)_{z=\eta}=\phi_{0x}-\frac{1}{2!}d^2\phi_{0xxx}+\frac{1}{4!}d^4\phi_{0xxxxx}+\cdots \tag{1.8}$$

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=\eta}=-d\phi_{0xx}+\frac{1}{3!}d^3\phi_{0xxxx}+\cdots \tag{1.9}$$

where $d=h+\eta$. From (1.7) we can get (see Zou, 1999).

$$u_0 = \overline{u}_p + \frac{1}{3!} d^2 \overline{u}_{pxx} + \frac{7}{3 \cdot 5!} d^4 \overline{u}_{pxxxx} + O(\mu^6) \tag{1.10}$$

where $u_0 = \phi_{0x}$, Substituting (1.10) into (1.8) and (1.9) gives

$$\left(\frac{\partial \phi}{\partial x}\right)_{z=\eta} = \overline{u}_p - \frac{1}{3} d^2 \overline{u}_{pxx} - \frac{1}{45} d^4 \overline{u}_{pxxxx} + \cdots \tag{1.11}$$

$$w_\eta = \left(\frac{\partial \phi}{\partial z}\right)_{z=\eta} = -d\overline{u}_{px} + \frac{1}{3!} d^3 \overline{u}_{pxxx} + \cdots \tag{1.12}$$

Then u_η in (1.4) can be expressed as

$$u_\eta = \overline{u} + \hat{u}_r + \hat{u}_p \tag{1.13a}$$

where

$$\hat{u}_p = \left(\frac{\partial \phi}{\partial x}\right)_{z=\eta} - \overline{u}_p = -\frac{1}{3} d^2 \overline{u}_{pxx} - \frac{1}{45} d^4 \overline{u}_{pxxxx} + \cdots \tag{1.13b}$$

$$\hat{u}_r = U(\eta) - \overline{u}_r \tag{1.13c}$$

Substituting (1.13) and (1.12) into (1.3) for u_η and w_η, then differentiating the resultant equation with respect to x, applying the following relation for $\partial \Psi_\eta / \partial x$ (see Appendix)

$$\Omega \frac{\partial \Psi_\eta}{\partial x} = -\Omega \frac{\partial \eta}{\partial t} = -\frac{\partial U(\eta)}{\partial t} \tag{1.14}$$

and replacing $\overline{u}_p$ by $\overline{u} - \overline{u}_r$ will give the following equations accurate to $O(\varepsilon\mu^2, \mu^4)$

$$\overline{u}_t + \overline{u}\,\overline{u}_x + g\eta_x + G + \Delta M = R + \Delta P \tag{1.15a}$$

Where

$$G = -\frac{1}{3} d^2 \overline{u}_{xxt} + [\frac{1}{2} d^2 (\overline{u}_x)^2 - \frac{1}{3} d^2 \overline{u}\,\overline{u}_{xx}]_x + \frac{1}{3} d\eta_t \overline{u}_{xx} - d\eta_x \overline{u}_{xt} \tag{1.15b}$$

$$R = \frac{1}{45} d^4 \overline{u}_{xxxxt} \tag{1.15c}$$

$$\Delta M = \hat{u}_r \hat{u}_{rx} - \frac{1}{3} [d^2 \hat{u}_r (\overline{u} - \overline{u}_r)_{xx}]_x \tag{1.15d}$$

$$\Delta P = \hat{u}_{rt} + (\hat{u}_r \overline{u})_x + \frac{1}{3} d^2 \overline{u}_{rxxt}$$

$$+ [\frac{1}{2} d^2 (\overline{u}_{rx})^2 + \frac{1}{3} d^2 \overline{u}\overline{u}_{rxx}]_x - (d^2 \overline{u}_x \overline{u}_{rx})_x$$

$$- \frac{1}{3} d\eta_t \overline{u}_{rxx} + d\eta_x \overline{u}_{rxt} - \frac{1}{45} d^4 \overline{u}_{rxxxxt} \quad (1.15e)$$

Another governing equation is the continuity equation, which has the usual form in term of $\overline{u}$

$$\eta_t + (d\overline{u})_x = 0 \quad (1.16)$$

From (1.1), (1.6c) and (1.16), we have

$$\hat{u}_r = U(\eta) - \overline{u}_r = \frac{1}{2} \Omega d \quad (1.17)$$

$$\hat{u}_{rt} = \frac{1}{2} \Omega d_t = -\frac{1}{2} \Omega (d\overline{u})_x = -(\overline{u}_r \overline{u})_x \quad (1.18)$$

So we have the relation

$$\hat{u}_{rt} + (\hat{u}_r \overline{u})_x = 0 \quad (1.19)$$

Substituting this into (1.5e) give the following simpler expression for ΔP

$$\Delta P = \frac{1}{3} d^2 \overline{u}_{rxxt} + [\frac{1}{2} d^2 (\overline{u}_{rx})^2 + \frac{1}{3} d^2 \overline{u}\overline{u}_{rxx}]_x - (d^2 \overline{u}_x \overline{u}_{rx})_x$$

$$- \frac{1}{3} d\eta_t \overline{u}_{rxx} + d\eta_x \overline{u}_{rxt} - \frac{1}{45} d^4 \overline{u}_{rxxxxt} \quad (1.20)$$

The dispersion property

Now we analyze the dispersion property of the equations derived above, we assume the order of magnitude of Ω is $O(1)$ and linearize the equation. After substituting the expressions for $\overline{u}_r$ and $\hat{u}_r$ into the equation (1.15), we get the following linearized expressions for the nonlinear terms in the equation.

$$\overline{u}\overline{u}_x \approx \frac{1}{2} U_0 (\frac{1}{2} \Omega \eta_x + \overline{u}_{px}) \quad (1.21)$$

$$\Delta M \approx \frac{1}{4} \Omega U_0 \eta_x - \frac{1}{6} U_0 h^2 \overline{u}_{pxxx} \quad (1.22)$$

$$G-\Delta P-R\approx\frac{1}{2}\Omega\frac{D\eta}{Dt}+\frac{1}{2}U_0\overline{u}_{px}$$

$$-\frac{1}{3}h^2\overline{u}_{pxxt}-\frac{1}{6}U_0h^2\overline{u}_{pxxx}-\frac{1}{45}h^2\overline{u}_{pxxxxt} \quad (1.23)$$

in which

$$U_0=U(0)=\Omega h \quad (1.24)$$

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x} \quad (1.25)$$

Here U_0 is the current velocity at z=0. In the above derivation, (1.15b) has been used for P, i.e. the first two terms in (1.15b) remain although they can cancel each other, see (1.19). This can make us easily to get the dispersion relation in the form expressed by the differentiation operator (1.25). To this end, we utilized the following expression for these two terms.

$$\hat{u}_{rt}+(\hat{u}_r\overline{u})_x=\frac{1}{2}\Omega\frac{D\eta}{Dt}+\frac{1}{2}U_0\overline{u}_{px} \quad (1.26)$$

From the above results, (1.15a) and (1.16) can be linearized as

$$\frac{D\overline{u}_p}{Dt}+g\eta_x+\Omega\frac{D\eta}{Dt}=\frac{1}{3}h^2\frac{D\overline{u}_{pxx}}{Dt}+\frac{1}{45}h^4\frac{D\overline{u}_{pxxxx}}{Dt} \quad (1.27)$$

and

$$\frac{D\eta}{Dt}+h\overline{u}_{px}=0 \quad (1.28)$$

In the last term of (1.27), $\partial/\partial t$ has been replaced by D/Dt and the error of this is $O(\varepsilon\mu^4)$, which can be ignored up to the accuracy of the equations. Eliminating η from the above two equations yields

$$\frac{D^2\overline{u}_p}{Dt^2}-gh\overline{u}_{pxx}-U_0\frac{D\overline{u}_{px}}{Dt}=\frac{1}{3}h^2\frac{D^2\overline{u}_{pxx}}{Dt^2}+\frac{1}{45}h^4\frac{D^2\overline{u}_{pxxxx}}{Dt^2} \quad (1.29)$$

Assume

$$\overline{u}_p(x,t)=e^{i(k_0x-\omega_0t)} \quad (1.30)$$

k_0 and ω_0 are the wave number and frequency respectively. Substituting (1.30) into (1.29) gives

$$(1+\frac{1}{3}k_0^2h^2-\frac{1}{45}k_0^4h^4)\omega^2-ghk_0^2(1+\frac{\Omega}{gk_0}\omega)=0 \quad (1.31a)$$

where

$$\omega=\omega_0-k_0U_0 \quad (1.31b)$$

(1.31a) can also be written as

$$\omega^2 - ghk_0^2(1+\frac{\Omega}{gk_0}\omega)(1-\frac{1}{3}k_0^2h^2+\frac{2}{15}k_0^4h^4-\cdots)=0 \qquad (1.32)$$

This is the dispersion relation of the present equations. By applying the following expansion

$$\text{th}\, kh = kh - \frac{1}{3}(kh)^3 + \frac{2}{15}(kh)^5 + \cdots \qquad (1.33)$$

We know that this relation agrees with the result accurate to $O(k^4h^4)$ of the following accurate dispersion relation

$$\omega^2 - gk_0[1+\frac{\Omega}{gk_0}\omega]\,\text{th}\, k_0 h = 0 \qquad (1.34a)$$

Here

$$\omega = \omega_0 - k_0 U_0 \qquad (1.34b)$$

Extension to horizontal two dimensions

The equations derived in the last section can be extended to horizontal two dimension. Suppose the flow is rotational and has the vorticity

$$\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z) = \nabla' \times \boldsymbol{V} \qquad (2.1)$$

in which $\boldsymbol{V} = (u, v, w)$, ∇' is the three dimensional gradient operator: $\nabla' = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. The expression (2.1) has the following components:

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = -\omega_z, \quad \frac{\partial \boldsymbol{u}}{\partial z} - \mu^2 \nabla w = \boldsymbol{s} \qquad -h < z < \varepsilon\eta \qquad (2.2a,b)$$

Here $\boldsymbol{s}$ is the transformed horizontal vorticity defined by

$$\boldsymbol{s} = (\omega_x, \omega_y)T = (\omega_y, -\omega_x) \qquad (2.3)$$

in which T is the transformation matrix defined by

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad (2.4)$$

∇ in (2.2b) is the horizontal two dimensional gradient operator $\nabla = (\partial/\partial x, \partial/\partial y)$.

We can decompose the solution into the potential part $\boldsymbol{u}_p, w_p$ and the rotational part $\boldsymbol{u}_r$, w_r:

$$\boldsymbol{u} = \boldsymbol{u}_p + \boldsymbol{u}_r \qquad w = w_p + w_r \tag{2.5a,b}$$

The solutions of the two parts can be sought separately. The potential part satisfies the following equations and boundary condition:

$$\frac{\partial w_p}{\partial z} + \nabla \cdot \boldsymbol{u}_p = 0, \quad -h < z < \varepsilon\eta \tag{2.6a}$$

$$\frac{\partial \boldsymbol{u}_p}{\partial z} - \mu^2 \nabla w_p = 0, \quad -h < z < \varepsilon\eta \tag{2.6b}$$

$$w_p = -\nabla h \cdot \boldsymbol{u}_p, \qquad z = -h \tag{2.6c}$$

The rotational part satisfies the following equations and boundary condition:

$$\frac{\partial w_r}{\partial z} + \nabla \cdot \boldsymbol{u}_r = 0, \quad -h < z < \varepsilon\eta \tag{2.7a}$$

$$\frac{\partial \boldsymbol{u}_r}{\partial z} - \mu^2 \nabla w_r = \boldsymbol{s}, \quad -h < z < \varepsilon\eta \tag{2.7b}$$

$$\boldsymbol{u}_r = w_r = 0 \qquad z = -h \tag{2.7c}$$

In the following, we will not give the solution for $\boldsymbol{u}_p$, which has been know and only gives the solutions for $\boldsymbol{u}_r$ and w_r. They have the following form of expressions accurate to $O(\mu^6)$

$$\boldsymbol{u}_r = \boldsymbol{u}_r^{(0)} - \nabla\nabla \cdot \int_{-h}^{z}\int_{-h}^{z} \boldsymbol{u}_r^{(0)}\, \mathrm{d}z^2$$

$$+ \nabla\nabla \cdot \nabla\nabla \cdot \int_{-h}^{z}\int_{-h}^{z}\int_{-h}^{z}\int_{-h}^{z} \boldsymbol{u}_r^{(0)}\, \mathrm{d}z^4 + O(\mu^6) \tag{2.8a}$$

$$w_r = -\nabla \cdot \int_{-h}^{z} \boldsymbol{u}_r^{(0)}\, \mathrm{d}z + \nabla \cdot \nabla\nabla \cdot \int_{-h}^{z}\int_{-h}^{z}\int_{-h}^{z} \boldsymbol{u}_r^{(0)}\, \mathrm{d}z^3 + O(\mu^4) \tag{2.8b}$$

in which

$$\boldsymbol{u}_r^{(0)} = \int_{-h}^{z} \boldsymbol{s}\, \mathrm{d}z \tag{2.9}$$

Substituting these expressions for velocity into Eular Equations, we can get the following form of the higher order Boussinesq equations with the inclusion of the vorticity

$$\eta_t + \nabla \cdot (d\bar{\boldsymbol{u}}) = 0 \tag{2.10a}$$

$$\overline{\boldsymbol{u}}_t + (\overline{\boldsymbol{u}} \cdot \nabla)\overline{\boldsymbol{u}} + g\nabla\eta + \boldsymbol{G} + \Delta\boldsymbol{M} = \boldsymbol{R} + \Delta\boldsymbol{P} \tag{2.10b}$$

in which

$$\boldsymbol{G} = -\{d\,\mathrm{D}[\frac{1}{3}\nabla(d\nabla\cdot\overline{\boldsymbol{u}}) + \frac{1}{3}\nabla(\nabla h\cdot\overline{\boldsymbol{u}})] + \frac{1}{6}\nabla h\,\mathrm{D}(d\nabla\cdot\overline{\boldsymbol{u}}) + \nabla\eta\,\mathrm{D}[\frac{2}{3}d\nabla\cdot\overline{\boldsymbol{u}} + \nabla h\cdot\overline{\boldsymbol{u}}]\} \tag{2.11}$$

$$\boldsymbol{R} = -\frac{1}{24}h^3\nabla\nabla\cdot\nabla\nabla\cdot(h\overline{\boldsymbol{u}}_p)_t + \frac{1}{12}h^2\nabla\nabla\cdot[h\nabla\nabla\cdot(h\overline{\boldsymbol{u}}_p)_t] + \frac{1}{120}h^4\nabla\nabla\cdot\nabla\nabla\cdot(\overline{\boldsymbol{u}}_p)_t - \frac{1}{36}h^2\nabla\nabla\cdot[h^2\nabla\nabla\cdot(\overline{\boldsymbol{u}}_p)_t] \tag{2.12}$$

$$\Delta\boldsymbol{M} = \int_{-1}^{0}\left(\hat{\boldsymbol{u}}_{\mathbf{r}}\cdot\overline{\nabla}\right)\hat{\boldsymbol{u}}_{\mathbf{r}}\,\mathrm{d}Z + \frac{1}{d}\int_{-1}^{0}\hat{\boldsymbol{u}}_{\mathbf{r}}\overline{\nabla}\cdot(d\hat{\boldsymbol{u}}_{\mathbf{r}})\mathrm{d}Z \tag{2.13}$$

$$\Delta\boldsymbol{P} = \frac{1}{d}\nabla\cdot\{d^2\,\mathrm{D}[\nabla\cdot(d(\overline{\mathbf{P}}_2^0 - \overline{\mathbf{P}}_1^0)) - \nabla\eta\cdot(\overline{\mathbf{P}}_1^0 - \overline{\mathbf{P}}_0^0) - \nabla d\cdot(\overline{\mathbf{P}}_1^1 - \overline{\mathbf{P}}_0^1)]\} \;\; -\nabla h\,\mathrm{D}[\nabla\cdot(d\overline{\mathbf{P}}_1^0) - \nabla\eta\cdot\overline{\mathbf{P}}_0^0 - \nabla d\cdot\overline{\mathbf{P}}_0^1] \tag{2.14}$$

Here $d = h + \eta$, the other definitions are as followings

$$\mathrm{D} = \frac{\partial}{\partial t} + \varepsilon\overline{\boldsymbol{u}}\cdot\nabla\,; \quad \overline{\nabla} = \nabla + (\varepsilon\nabla\eta + \frac{z-\eta}{d}\nabla d)\frac{\partial}{\partial z} \tag{2.15a,b}$$

$$\hat{\boldsymbol{u}}_r = \boldsymbol{u}_r - \overline{\boldsymbol{u}}_r\,, \quad \boldsymbol{u}_r \approx \int_{-h}^{z}\mathbf{s}\,\mathrm{d}z\,, \quad \overline{\boldsymbol{u}}_r = \frac{1}{d}\int_{-h}^{\eta}\boldsymbol{u}_r\,\mathrm{d}z \tag{2.16a,b,c}$$

$$\mathbf{P}_m^n = \underbrace{\int_{-1}^{Z}\mathrm{d}Z\int_{-1}^{Z}\mathrm{d}Z\cdots\int_{-1}^{Z}}_{m} Z^n\hat{\boldsymbol{u}}_r\mathrm{d}Z\,, \quad \overline{\mathbf{P}}_m^n = \int_{-1}^{0}\mathbf{P}_m^n\,\mathrm{d}Z \tag{2.17a,b}$$

From the above definitions, we have

$$\mathbf{P}_0^0 = \hat{\boldsymbol{u}}_r\,, \qquad \mathbf{P}_0^1 = \hat{\boldsymbol{u}}_r Z \tag{2.18a,b}$$

Since $\hat{\boldsymbol{u}}_r$ is produced by the vertical variation of the rotational velocity, the term $\Delta\boldsymbol{M}$ is the contribution to the convection terms from the vertical variation of the rotational velocity and $\Delta\boldsymbol{P}$ is the contribution to the pressure terms from the vertical variation of the rotational velocity.

It is noticed that the term $\Delta\boldsymbol{M}$ is expressed in the transformed coordinates :

$$X = x, \quad Y = y, \quad Z = \frac{z-\eta}{d}, \quad T = t \tag{2.19}$$

where $d = h + \eta$. In the new coordinates, the fluid domain $-h < z < \varepsilon\eta$ is transformed into the fixed domain: $-1 < Z \le 0$. The relations between the derivatives in the old and the new coordinates are

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial T}-\frac{1}{d}(\varepsilon\eta_T+Zd_T)\frac{\partial}{\partial Z}$$
$$\nabla=\overline{\nabla}-\frac{1}{d}(\varepsilon\overline{\nabla}\eta+Z\overline{\nabla}d)\frac{\partial}{\partial Z} \qquad (2.20)$$
$$\frac{\partial}{\partial z}=\frac{1}{d}\frac{\partial}{\partial Z}$$

Using this new coordinates make the term $\Delta \boldsymbol{M}$ in a compact form. If this term is expressed in the original coordinates, it will be of a more complicated form by noticing the relation (2.15b).

The terms related to η and ∇d in (2.13) reflect the nonlinear and bottom effects on the convection terms $\Delta \boldsymbol{M}$. These effects also exit for the pressure term $\Delta \boldsymbol{P}$, but contrary to (2.13), they are expressed explicitly in the expression (2.14) by the terms related to η and ∇d. Neglecting these terms and other nonlinear terms can make the expression for $\Delta \boldsymbol{P}$ simpler as shown below.

$$\Delta \boldsymbol{P}=\frac{1}{d}\nabla^2\cdot[d^3(\overline{\mathbf{P}}_2^0-\overline{\mathbf{P}}_1^0)]_t+O(\varepsilon\sigma)$$
$$=\frac{1}{d}\nabla^2\cdot[\int_{-h}^{\eta}\mathrm{d}z\int_{z}^{\eta}\mathrm{d}\,z\int_{-h}^{z}\hat{\boldsymbol{u}}_r\,\mathrm{d}\,z]_t+O(\varepsilon\sigma) \qquad (2.21)$$

Analysis of the equations

In order to check the equations derived above, we consider the one dimensional form of the equations, the term ΔM in (2.13) becomes for this case

$$\Delta M=\frac{1}{d}\int_{-1}^{0}[d\hat{u}_r(\hat{u}_r)_X+\hat{u}_r(d\hat{u}_r)_X]\mathrm{d}\,Z$$
$$=\frac{1}{d}\int_{-1}^{0}[d(\hat{u}_r)^2]_X\,\mathrm{d}\,Z=\frac{1}{d}[\int_{-1}^{0}d(\hat{u}_r)^2\,\mathrm{d}\,Z]_X \qquad (3.1)$$

Applying the transformation (2.20) gives

$$\Delta M=\frac{1}{d}[\int_{-h}^{\eta}(\hat{u}_r)^2\,\mathrm{d}\,z]_x=\frac{1}{d}[\int_{-h}^{\eta}(u_r^2-\overline{u}_r^2)\mathrm{d}\,z]_x \qquad (3.2)$$

The pressure term $\Delta \boldsymbol{P}$ in (2.22) becomes

$$\Delta P=\frac{1}{d}\frac{\partial^2}{\partial x^2}[\int_{-h}^{\eta}\mathrm{d}z\int_{z}^{\eta}\mathrm{d}\,z\int_{-h}^{z}\hat{u}_r\,\mathrm{d}\,z]_t$$
$$=\frac{1}{d}\frac{\partial^2}{\partial x^2}[\int_{-h}^{\eta}\mathrm{d}z\int_{z}^{\eta}\mathrm{d}\,z\int_{-h}^{z}(u_r-\overline{u}_r)\mathrm{d}\,z]_t \qquad (3.3)$$

Substituting (3.2) and (3.3) into (2.10b) in one dimensional form and neglecting the terms R of $O(\mu^4)$ yields

$$\bar{u}_t + \bar{u}\bar{u}_x + G + \frac{1}{d}\left[\int_{-h}^{\eta}(u_r^2 - \bar{u}_r^2)\,\mathrm{d}z\right]_x = \frac{1}{d}\left[\int_{-h}^{\eta}\mathrm{d}z\int_{z}^{\eta}\mathrm{d}z\int_{-h}^{z}(u_r - \bar{u}_r)_t\,\mathrm{d}z\right]_{xx} \tag{3.4}$$

or

$$(d\bar{u})_t + \left[d\bar{u}^2 + \int_{-h}^{\eta}(u_r^2 - \bar{u}_r^2)\,\mathrm{d}z\right]_x + dG = \left[\int_{-h}^{\eta}\mathrm{d}z\int_{z}^{\eta}\mathrm{d}z\int_{-h}^{z}(u_r - \bar{u}_r)\,\mathrm{d}z\right]_{xxt} \tag{3.5}$$

This is the same as the equations (19) in the paper of Veeramony and Svendsen (2000) .

The determination of horizontal vorticity for breaking waves

In solving the equations, a vertical distribution of the horizontal vorticity expressed by s needs to be given. For vertical two dimensional flow, Veeramony and Svendsen (2000) determine the vertical distribution of vorticity by solving the vorticity transport equation. For the three dimensional flow concerned here, it is difficult to adopt this kind of methods directly due to the time-consuming and the complicate boundary condition for the vorticity on the free surface in the numerical solution of the three dimensional vorticity transport equation. So an approximation method needs to be introduced in order to get the closure of the equations derived above. This will be investigated in the future researches

Summary

The higher order Boussinesq equations with the inclusion of vorticity are investigated. First, the equations are derived in details for the horizontal one-dimension and constant vorticity case. Then, the extension to the two-dimensional case is given. These derivations show the structure the Boussinesq equations will have when the whole vorticity field is considered. The equations are expressed in term of the depth averaged velocity and this is preferable since this form of equation have the property of conserving the potential vorticity ω_z / d (ω_z is the vertical component of vorticity), other form of the equations, such as those in term of the velocity at arbitrary level, does not have this property.

The dispersion property of the derived equations is discussed and it agrees with the shallow water approximation of the accurate dispersion relation for the wave propagating in uniform shear currents.

For the closure of the equations, a vertical distribution of horizontal vorticity needs to be determined in the future researches

Acknowledgments

The first author would like to thank the financial supports by the National Natural Science Founding of China, under Grant No. 59979002 and Grant No. 5047053. The work of J. Kirby and F. Shi was supported by the Office of Naval Research and the National Ocean Partners Program.

References

Deigaard, R. and J. Fredsoe (1989). Shear stress distribution in dissipative water waves. Coastal Engng., 13:357-378.

Karamabas, T. K. and C. Koutitas (1992). A breaking wave propagation model based on the Boussinesq equations. Coastal Engng. 18:1-19.

Madsen, P. A., O. R. Sorensen and H. A. Schaffer (1997). Surf zone dynamics simulated by a Boussinesq type model. Part I. Model description and cross shore motion of regular waves. Coastal Engng. 32:255-287.

Peregrine D.H. 1995.Vorticity and eddies in the surf zone, Coastal Dynamics'95, Gdansk, Poland, 461-464.

.Peregrine D.H.1998.Vorticity and eddies in the surf zone. Theoretical and Computational Fluid Dynamics,10:295-309.

Rego, V. S. , J. T. Kirby and D. Thompson (2001). Boussinesq waves on vertically sheared currents. Proc. 4th Intl. Symp. Ocean Wave Measurement and Analysis, Waves 2001, San Francisco, September 2-6, 904-913.

Schaffer, H. A., P.,A., Madsen and R. Deigaard (1993). A Boussinesq model for wave breaking in shallow water. Coastal Engng., 20:185-202.

Shen, C. Y. (2000). Constituent Boussinesq equations for waves and currents. J. Physical Oceangraphy, 31:850-859.

Veeramony, J. and I. A. Svendsen (2000). The flow in surf-zone waves. Coastal Engng., 39:93-122.

Wei, G. and J. T. Kirby (1995). A time dependent numerical code for extended Boussinesq equations. ASCE J. Waterway, Port, Coastal and Ocean Engng., 120:251-261.

Yu, K and I. A. Svendsen (1995). Breaking waves in surf zone. Coastal Dynamics'95, Gdansk, Poland, 329-340.

Zelt, J. A. (1991). The run-up of nonlinearbreaking and breaking solitary waves. Coastal Engng., 15:205-246.

Zou. Z. L. (1999). A higher order Boussinesq equations. Ocean Engng., 26:767-792.

Appendix

For the velocity potential ϕ on free surface

$$\phi_\eta = \phi(x,y,z,t)\big|_{z=\eta} = \phi(x,y,\eta,t) \tag{A.1}$$

we have

$$\left(\frac{\partial \phi}{\partial t}\right)_{z=\eta} = \frac{\partial \phi_\eta}{\partial t} - \left(\frac{\partial \phi}{\partial z}\right)_{z=\eta} \frac{\partial \eta}{\partial t} \tag{A.2}$$

$$(\nabla \phi)_{z=\eta} = \nabla \phi_\eta - \left(\frac{\partial \phi}{\partial z}\right)_{z=\eta} \nabla \eta \tag{A.3}$$

here $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. Substituting these relations into the boundary condition:

$$\frac{\partial \eta}{\partial t} + (U(\eta) + \nabla \phi_\eta) \cdot \nabla \eta = \left(\frac{\partial \phi}{\partial z}\right)_{z=\eta} \tag{A.4}$$

gives

$$\frac{\partial \eta}{\partial t} + (U(\eta) + \nabla \phi_\eta) \cdot \nabla \eta = (1 + \nabla \eta \cdot \nabla \eta)\left(\frac{\partial \phi}{\partial z}\right)_{z=\eta} \tag{A.5}$$

So for the stream function Ψ, we have

$$\begin{aligned} \frac{\partial \Psi_\eta}{\partial x} &= \left(\frac{\partial \Psi}{\partial x}\right)_{z=\eta} + \left(\frac{\partial \Psi}{\partial z}\right)_{z=\eta} \frac{\partial \eta}{\partial x} \\ &= -\left(\frac{\partial \phi}{\partial z}\right)_{z=\eta} + \left(U + \frac{\partial \phi}{\partial x}\right)_{z=\eta} \frac{\partial \eta}{\partial x} \\ &= -[1 + (\frac{\partial \eta}{\partial x})^2]\left(\frac{\partial \phi}{\partial z}\right)_{z=\eta} + [U(\eta) + \frac{\partial \phi_\eta}{\partial x}]\frac{\partial \eta}{\partial x} \end{aligned} \tag{A.6}$$

Applying (A.5) to (A.6) yields

$$\frac{\partial \Psi_\eta}{\partial x} = -\frac{\partial \eta}{\partial t} \tag{A.7}$$