Dispersive effects on wave-current interaction and vorticity transport in nearshore flows

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Frequency dispersive effects on the interaction of waves and currents in a nearshore circulation system are analyzed. By means of both analytical and numerical calculations, we find that dispersive effects are important in the description of the correct amount of the wave forcing, represented, within the generalized-Lagrangian-mean-like approach, by the pseudomomentum $\mathbf{p}$. They are particularly important when describing the flow using depth-averaged velocities. For some configurations, the depth-averaged current is forced not only by the nondispersive terms but also, with the same intensity, by the dispersive forcing terms. In such terms is included a vortex-force dispersive contribution, usually negligible in a small-wave amplitude approximation, which arises because of the presence of dissipative terms. © *2008 American Institute of Physics*.
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I. INTRODUCTION

A proper description of wave-current interaction is fundamental for any correct modeling of nearshore flow circulation. The interaction process can be studied in models that resolve the circulation at different time scales: the time scale of the currents (i.e., wave-averaged models) and the wave time scale (i.e., wave-resolving models).

When using wave-averaged approaches, an *a priori* wave forcing for the currents is needed. Many of the existing nearshore wave-averaged circulation models are based on nonlinear shallow water equations, written for depth- and wave-averaged velocities, forced by the wave radiation stress, and neglecting frequency-dispersion effects. The radiation stress was defined by Longuet–Higgins and Stewart[1] as the net flux of momentum due to the wave field. In particular, they ascribed changes in the mean flow to gradients in the radiation stress, conserving the total momentum of the combined system of waves and currents. With a similar approach, Yu and Slinn[2] analyzed the wave-current interaction in a rip current system. Recently, more attention has been paid to properly describe the wave forcing (i.e., McWilliams *et al.*[3] and Shi *et al.*[4]). In addition to the classical radiation stress formulation, a Craik–Leibovich vortex-force formulation, hereinafter CL vortex-force (i.e., Craik and Leibovich[5]), has been studied. Shi *et al.*,[4] using a quasi-3D nearshore circulation model developed by Svendsen *et al.*,[6] described and compared the CL-vortex type of forcing with the classical radiation stress approach.

On the other hand, on the wave time scale, a wave-resolving model allows us to implicitly consider the wave forcing of the mean currents, solving the flow without any distinction between waves and currents. Bühler and Jacobson,[7] hereinafter BJ01, investigated the wave-driven currents and the vorticity dynamics using wave-resolving nonlinear shallow water equations (NSWE) combined with a generalized-Lagrangian-mean (GLM) framework. Extending their work by taking into account the effects due to frequency dispersion, we look for an improvement in the description of the wave-current interaction and vorticity evolution. In the present study, we use a Boussinesq model that includes dispersive terms instead of using the NSWE model. Chen *et al.*[8] presented a Boussinesq model in which the vertical vorticity is conserved up to the second order in frequency dispersion $\mu=h_0/l_0$ (h_0 and l_0 being the scales for the offshore water depth and the wavelength, respectively), i.e., consistent with the order of approximation for the wave motion. A further improvement has recently been added by Chen,[9] hereinafter Ch06, who proposed fully nonlinear, potential vorticity-conserving Boussinesq-type equations. Such conservation is desirable for modeling wave-induced nearshore circulation, which, as described in Peregrine,[10] BJ01, Kennedy *et al.*,[11] Terrile *et al.*,[12] and Terrile and Brocchini,[13] is closely related to the vorticity dynamics.

The theoretical treatment of the present problem is thus twofold. First, we present the Boussinesq model of Ch06, which is modified to suit our purposes, i.e., we redefine the velocity vector taking into account second-order effects of vertical vorticity. Second, we analyze the dynamics governed by this model using a GLM approach similar to that of BJ01. The use of a Lagrangian description allows us to discuss the nonlinear dynamics in terms of the total Lagrangian transport, the wave pseudomomentum, and the corresponding quasi-Eulerian current. These quantities all appear naturally in wave-mean flow interaction problems.[14]

The outline of the paper is as follows. The Boussinesq model is presented Sec. II. In Sec. III, we investigate the influence of wave dispersion on the mean flow using a GLM approach, and we present numerical results for a rip current system. Section IV contains a discussion of the results and some concluding remarks.

II. NEARSHORE CIRCULATION AND DISPERSIVE EFFECTS

The standard Boussinesq equations for variable water depth in the Eulerian framework were first derived by Peregrine,[15] who used the depth-averaged velocity as a dependent variable. Recently various sets of fully nonlinear Boussinesq-type equations have been made available which differ in their description of vertical vorticity contributions at $O(\mu^2)$. Kirby[16] provides a discussion of the correction provided by Chen *et al.*[8] over the model of Wei *et al.*[17] However, Ch06 has shown that this correction is incomplete because the corrected equations are still only first-order accurate with respect to the conservation of potential vorticity. In the same work, he introduced an improved set of Boussinesq-type equations that are potentially vorticity-conserving.

In Ch06, both the continuity and the momentum equations are written in terms of a reference horizontal velocity vector $\mathbf{u}_\alpha=(u_\alpha,v_\alpha)$ at some reference elevation in the fluid layer $z=z_\alpha(x,y)=\alpha h_s(x,y)$. Here h_s represents the still-water depth while the parameter α is determined by fitting the linear dispersion relation of the Boussinesq model to Stokes-type solutions. The continuity equation is identical to that of Wei *et al.*[17] and reads

$$\frac{\partial \eta}{\partial t}+\nabla\cdot\mathbf{M}_\alpha=O(\mu^4\sqrt{gh_0}),\tag{1}$$

where

$$\mathbf{M}_\alpha=(h_s+\eta)\left[\mathbf{u}_\alpha+\mu^2\left\{\left(z_\alpha-\frac{1}{2}(\eta-h_s)\right)\nabla(\nabla\cdot(h_s\mathbf{u}_\alpha))+\left(\frac{z_\alpha^2}{2}-\frac{1}{6}(\eta^2-\eta h_s+h_s^2)\right)\nabla(\nabla\cdot\mathbf{u}_\alpha)\right\}\right],\tag{2}$$

h_s is the still water depth, and η is the free surface elevation. The associated momentum conservation equation given in Ch06 reads

$$\frac{\partial \mathbf{u}_\alpha}{\partial t}+(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha+g\nabla\eta+\mu^2(\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3)=O\left(\mu^4\frac{gh_0}{l_0}\right),\tag{3}$$

where

$$\mathbf{V}_1=\frac{z_\alpha^2}{2}\nabla(\nabla\cdot\mathbf{u}_{\alpha,t})+z_\alpha\nabla[\nabla\cdot(h_s\mathbf{u}_{\alpha,t})]-\nabla\left[\frac{\eta^2}{2}\nabla\cdot\mathbf{u}_{\alpha,t}+\eta\nabla\cdot(h_s\mathbf{u}_{\alpha,t})\right],\tag{4}$$

$$\mathbf{V}_2=\nabla\Big\{(z_\alpha-\eta)(\mathbf{u}_\alpha\cdot\nabla)[\nabla\cdot(h_s\mathbf{u}_\alpha)]+\tfrac{1}{2}(z_\alpha^2-\eta^2)(\mathbf{u}_\alpha\cdot\nabla)(\nabla\cdot\mathbf{u}_\alpha)\Big\}+\tfrac{1}{2}\nabla\{[\nabla\cdot(h_s\mathbf{u}_\alpha)+\eta\nabla\cdot\mathbf{u}_\alpha]^2\},\tag{5}$$

$$\mathbf{V}_3=(V_3^{(x)},V_3^{(y)}),\tag{6}$$

in which

$$V_3^{(x)}=-v_\alpha\omega_1-\omega_0\left\{\left[z_\alpha-\frac{1}{2}(\eta-h_s)\right]\frac{\partial}{\partial y}[\nabla\cdot(h_s\mathbf{u}_\alpha)]+\left[\frac{z_\alpha^2}{2}-\frac{1}{6}(\eta^2-\eta h_s+h_s^2)\right]\frac{\partial}{\partial y}(\nabla\cdot\mathbf{u}_\alpha)\right\},\tag{7}$$

$$V_3^{(y)}=u_\alpha\omega_1+\omega_0\left\{\left[z_\alpha-\frac{1}{2}(\eta-h_s)\right]\frac{\partial}{\partial x}[\nabla\cdot(h_s\mathbf{u}_\alpha)]+\left[\frac{z_\alpha^2}{2}-\frac{1}{6}(\eta^2-\eta h_s+h_s^2)\right]\frac{\partial}{\partial x}(\nabla\cdot\mathbf{u}_\alpha)\right\}.\tag{8}$$

Here the notation $(\,)_{,t}=\partial/\partial t(\,)$ has been used. The vertical component of the vorticity at the second order of approximation is given by the following expression:

$$\omega=\omega_0+\mu^2\omega_1+O\left(\mu^4\frac{\sqrt{gh_0}}{l_0}\right),\tag{9}$$

with

$$\omega_0=\frac{\partial v_\alpha}{\partial x}-\frac{\partial u_\alpha}{\partial y};\tag{10}$$

$$\omega_1=\frac{\partial z_\alpha}{\partial x}\left\{\frac{\partial}{\partial y}[\nabla\cdot(h_s\mathbf{u}_\alpha)]+z_\alpha\frac{\partial}{\partial y}(\nabla\cdot\mathbf{u}_\alpha)\right\}-\frac{\partial z_\alpha}{\partial y}\left\{\frac{\partial}{\partial x}[\nabla\cdot(h_s\mathbf{u}_\alpha)]+z_\alpha\frac{\partial}{\partial x}(\nabla\cdot\mathbf{u}_\alpha)\right\}.\tag{11}$$

In comparison with Wei *et al.*,[17] the only difference is the introduction of the second-order contribution of the vertical vorticity, $\mathbf{V}_3$, while the difference with respect to the more sophisticated model of Chen *et al.*[8] is the extra correction associated with ω_0 in $\mathbf{V}_3$. The implications of these corrections are connected with the conservation of the potential vorticity. Note that the vertical vorticity ω is depth uniform and depends on $\mathbf{u}_\alpha$, even if it is not its direct curl [i.e., $\omega\neq(\nabla\times\mathbf{u}_\alpha)\cdot\hat{\mathbf{k}}$ but $\omega_0=(\nabla\times\mathbf{u}_\alpha)\cdot\hat{\mathbf{k}}$].

To get a better understanding of the physical phenomena described by the equations of Ch06, and to obtain a "more transparent" set of equations, we introduce a new definition of the velocity vector, $\mathbf{u}_\omega$, which takes into account second-order effects of vertical vorticity. This leads to a velocity whose curl, projected along the vertical $\hat{\mathbf{k}}$, is exactly the complete vertical vorticity, ω, correct to the second order.

The horizontal vector velocity $\mathbf{u}_\omega$ is defined as a function of $\mathbf{u}_\alpha$ as

$$\mathbf{u}_\omega=\mathbf{u}_\alpha+\mu^2\mathbf{u}_{12},\tag{12}$$

with

$$\mathbf{u}_{12}=z_\alpha\nabla(\nabla\cdot(h_s\mathbf{u}_\alpha))+\frac{z_\alpha^2}{2}\nabla(\nabla\cdot\mathbf{u}_\alpha),\tag{13}$$

or, referring to the depth-averaged velocity $\hat{\mathbf{u}}$, as

$$\mathbf{u}_\omega=\hat{\mathbf{u}}+\mu^2\mathbf{u}_2,\tag{14}$$

with

$$\mathbf{u}_2 = \tfrac{1}{2}(\eta - h_s)\nabla(\nabla\cdot(h_s\mathbf{u}_\alpha)) + \tfrac{1}{6}(\eta^2 - \eta h_s + h_s^2)\nabla(\nabla\cdot\mathbf{u}_\alpha), \quad (15)$$

where $\mathbf{u}_2$ represents the difference, at order $O(\mu^2)$, between $\mathbf{u}_\omega$ and the depth-averaged horizontal velocity vector $\hat{\mathbf{u}}$. From Eq. (14), it immediately follows that $\omega=(\nabla\times\mathbf{u}_\omega)\cdot\hat{\mathbf{k}}+O(\mu^4\sqrt{gh_0}/l_0)$. Using the velocity $\mathbf{u}_\omega$, we rewrite the Boussinesq model of Ch06 as follows. The continuity equation becomes

$$\frac{\partial\eta}{\partial t} + \nabla\cdot\mathbf{M} = O(\mu^4\sqrt{gh_0}), \quad (16)$$

$$\mathbf{M} = (h_s+\eta)\Big[\mathbf{u}_\omega - \mu^2\big\{\tfrac{1}{2}(\eta - h_s)\nabla(\nabla\cdot(h_s\mathbf{u}_\omega)) + \tfrac{1}{6}(\eta^2 - \eta h_s + h_s^2)\nabla(\nabla\cdot\mathbf{u}_\omega)\big\}\Big] + O(\mu^4 h_0\sqrt{gh_0}). \quad (17)$$

The associated momentum equation reads

$$\frac{\partial\mathbf{u}_\omega}{\partial t} + (\mathbf{u}_\omega\cdot\nabla)\mathbf{u}_\omega + g\nabla\eta + \mu^2(\mathbf{V}_{12}^\omega + \mathbf{V}_3^\omega) = O\left(\mu^4\frac{gh_0}{l_0}\right), \quad (18)$$

where

$$\mathbf{V}_{12}^\omega = -\nabla\left[\frac{\eta^2}{2}\nabla\cdot\mathbf{u}_{\omega,t} + \eta\nabla\cdot(h_s\mathbf{u}_{\omega,t})\right] + \frac{1}{2}\nabla\{[\nabla\cdot(h_s\mathbf{u}_\omega) + \eta\nabla\cdot\mathbf{u}_\omega]^2\} + \nabla\Bigg\{(z_\alpha - \eta)(\mathbf{u}_\omega\cdot\nabla)[\nabla\cdot(h_s\mathbf{u}_\omega)] + \frac{1}{2}(z_\alpha^2 - \eta^2)(\mathbf{u}_\omega\cdot\nabla)(\nabla\cdot\mathbf{u}_\omega)\Bigg\}, \quad (19)$$

$$\mathbf{V}_3^\omega = \mathbf{u}_2\times(\nabla\times\mathbf{u}_\omega). \quad (20)$$

In the momentum equation, $\mathbf{V}_{12}^\omega$ gives an irrotational dispersive contribution, therefore, taking the curl of Eq. (18) leads to the following vorticity equation for ω:

$$\frac{\partial\omega}{\partial t} + (\mathbf{u}_\omega\cdot\nabla)\omega = -\omega\nabla\cdot\mathbf{u}_\omega - \mu^2\nabla\times\mathbf{V}_3^\omega + O\left(\mu^4\frac{gh_0}{l_0^2}\right), \quad (21)$$

where a *vortex-force* term appears (second term on the right-hand side). The potential vorticity equation, written for the transport velocity $\mathbf{u}_\omega$, reads

$$\frac{\partial\mathbf{q}}{\partial t} + (\mathbf{u}_\omega\cdot\nabla)\mathbf{q} = -\mu^2\{\nabla\times(\mathbf{u}_2\times\mathbf{q}) - \mathbf{q}\nabla\cdot\mathbf{u}_2\} + O\left(\mu^4\frac{gh_0}{l_0^3}\right), \quad (22)$$

in which

$$\mathbf{q}\equiv\frac{\nabla\times\mathbf{u}_\omega}{h} \quad (23)$$

and where $h=h_s+\eta$ represents the total water depth. Here we can identify the dispersion effects contributing to the potential vorticity $\mathbf{q}$, while moving with velocity $\mathbf{u}_\omega$. The first term on the right-hand side represents the forcing due to the *vortex-force* effect produced by the dispersive velocity contribution $\mathbf{u}_2$ and it corresponds to the term $-\mu^2\nabla\times\mathbf{V}_3^\omega$ in the vorticity equation (21). The second term on the right-hand side of Eq. (22) is a 2D stretching term due to the dispersive contribution.

Note that the potential vorticity conservation, obtained by Ch06, can still be obtained in terms of the depth-averaged transport velocity $\hat{\mathbf{u}}$. Combining Eqs. (14), (16), and (21), we get the following equation:

$$\frac{\partial\mathbf{q}}{\partial t} + (\hat{\mathbf{u}}\cdot\nabla)\mathbf{q} = O\left(\mu^4\frac{gh_0}{l_0^3}\right), \quad (24)$$

where the conservation of the potential vorticity, obtained in Eq. (24), is consistent with the level of approximation in the Boussinesq model for the pure wave motion, having accuracy up to order $O(\mu^2)$.

III. WAVE-CURRENT INTERACTION ANALYSIS AND RESULTS

In this section, we study the wave-current interaction in nearshore flows applying a GLM-like approach to the fully nonlinear Boussinesq model described in the preceding section. Following BJ01, we perform an asymptotic expansion in terms of a small-wave amplitude. Even if this assumption is rather restrictive, it provides insight into the physical processes and the influence of the frequency dispersive terms on the nearshore dynamics. The analysis is performed for very weakly dissipative waves, the dissipation being only due to subgrid mixing and to bed friction. Dissipative effects induced by wave breaking can lead to weakly dispersive waves inshore of the break point, hence masking the importance of dispersive contributions; however, we focus here on properly representing dispersion because (i) this is the fundamental effect which, balancing nonlinearities, allows for a proper prescription of the break point, and (ii) it quantitatively influences the wave forcing.

We are also interested in the forcing of the potential vorticity due to dispersion effects. For this reason, though we are employing a small-wave amplitude assumption, we start with analyzing the fully nonlinear Boussinesq model of the preceding section, which retains dispersive terms usually neglected in nonlinear shallow water models.

In the small-wave approximation, we consider the response of the currents to slowly varying, small-amplitude gravity waves. With a background state of rest of $O(1)$, we assume that the flow field can be decomposed into a mean part $\bar{\phi}$ of $O(a^2)$ (Eulerian mean is denoted with an overbar) and a disturbance part ϕ' of $O(a)$, where a is the wave amplitude, such that $\phi=\phi'+\bar{\phi}$ and $\overline{\phi'}=0$. The average is taken over the wave period. We define a generic velocity field $\mathbf{u}$ and the surface height η such that

$$\mathbf{u} = \mathbf{u}' + \bar{\mathbf{u}} + O(a^3) \quad \text{and } \eta = \eta' + \bar{\eta} + O(a^3), \tag{25}$$

where the disturbance quantities $\mathbf{u}'$ and η' are $O(a)$, while the mean quantities, $\bar{\mathbf{u}}$ and $\bar{\eta}$, the latter representing the mean surface elevation, are $O(a^2)$. Note that the still water depth h_s represents a background field and, therefore, is $O(1)$.

For the analysis we attended, the dissipative body force R, which appears in the momentum equation (18), is essentially made of weak subgrid mixing and bed friction dissipations. Substituting Eq. (25) in the continuity equations (16) and (17) and momentum equation (18), we obtain the following continuity and momentum to $O(a)$:

$$\frac{\partial \eta'}{\partial t} + \nabla \cdot \mathbf{M}' = 0, \tag{26}$$

$$\frac{\partial \mathbf{u}'_\omega}{\partial t} + g \nabla \eta' = \mathbf{R}', \tag{27}$$

with

$$\mathbf{M}' = h_s[\mathbf{u}'_\omega - \mu^2 \mathbf{u}'_2], \tag{28}$$

$$\mathbf{u}'_2 = -\frac{h_s}{2} \nabla(\nabla \cdot (h_s \mathbf{u}'_\omega)) + \frac{h_s^2}{6} \nabla(\nabla \cdot \mathbf{u}'_\omega) + O(a^2). \tag{29}$$

Following BJ01, we introduce a linear particle displacement $\boldsymbol{\xi}' = (\xi'_a, \xi'_b)$, with mean value $\overline{\boldsymbol{\xi}'} = 0$, such that

$$\frac{\partial \boldsymbol{\xi}'}{\partial t} = \hat{\mathbf{u}}'. \tag{30}$$

Using Eqs. (14) and (28), from Eqs. (26) and (30) we obtain

$$\eta' + \nabla \cdot (h_s \boldsymbol{\xi}') = 0. \tag{31}$$

A useful wave property we can now obtain, using the velocity field $\mathbf{u}'_\omega$ and η', is the Stokes drift $\bar{\mathbf{u}}^S_\omega$. This is usually defined as the difference between the Lagrangian mean velocity $\bar{\mathbf{u}}^L_\omega$ (representing the mean velocity of the particle whose *mean* position is $\mathbf{x}$ at time t) and the Eulerian mean velocity $\bar{\mathbf{u}}_\omega$. Note that the overbar denotes averaging over the fast time scales. The Stokes drift is found from[14,18]

$$\bar{\mathbf{u}}^S_\omega = \bar{\mathbf{u}}^L_\omega - \bar{\mathbf{u}}_w = \overline{(\boldsymbol{\xi}' \cdot \nabla)\mathbf{u}'_\omega}$$
$$= \frac{1}{h_s}\overline{(h_s \boldsymbol{\xi}' \cdot \nabla)\mathbf{u}'_\omega} = -\frac{1}{h_s}\overline{\nabla \cdot (h_s \boldsymbol{\xi}')\mathbf{u}'_\omega} \approx \frac{1}{h_s}\overline{\eta' \mathbf{u}'_\omega}. \tag{32}$$

Equation (32) has been obtained using Eq. (31) and neglecting terms of order $O(\varepsilon a^2)$. Here $\varepsilon \ll 1$ is the ratio between the horizontal gradient of the mean current and the horizontal gradient of the disturbance field. Expressions similar to Eq. (32) valid for Stokes correction, i.e., $\bar{\phi}^S = \bar{\phi}^L - \bar{\phi}$, can be found.

A description of the response of currents to the gravity waves is achieved analyzing the $O(a^2)$ continuity and momentum equations. This set of equations is obtained by time-averaging the Boussinesq model equations (16)–(18) over the fast time scale and retaining all terms up to $O(a^2)$. Using Eq. (25), we get (more details can be found in the Appendix)

$$\frac{\partial \bar{\eta}}{\partial t} + \nabla \cdot [h_s(\bar{\mathbf{u}}^L_\omega - \mu^2 \bar{\mathbf{u}}^L_2)] = 0, \tag{33}$$

$$\frac{\partial \bar{\mathbf{u}}_\omega}{\partial t} + \overline{(\mathbf{u}'_\omega \cdot \nabla)\mathbf{u}'_\omega} + g \nabla \bar{\eta} + \mu^2 \overline{\mathbf{V}^\omega_{12}} + \mu^2 \overline{\mathbf{u}'_2 \times (\nabla \times \mathbf{u}'_\omega)} = \bar{\mathbf{R}}. \tag{34}$$

Using Eq. (19), we note that $\mu^2 \overline{\mathbf{V}^\omega_{12}}$ in Eq. (34) is of $O(\varepsilon a^2)$ and can be neglected.

The advective term is handled in the same way as in BJ01, and it can be rewritten as

$$\overline{(\mathbf{u}'_\omega \cdot \nabla)\mathbf{u}'_\omega} = \frac{1}{h_s} \nabla \cdot \left(h_s \overline{\mathbf{u}'_\omega \mathbf{u}'_\omega} + \mathbf{I}\frac{g}{2}\overline{\eta'^2} \right) + \frac{\partial \bar{\mathbf{u}}^S_\omega}{\partial t}$$
$$- \mu^2 \frac{1}{h_s} \overline{\nabla \cdot (h_s \mathbf{u}'_2)\mathbf{u}'_\omega} - \frac{1}{h_s}\overline{\eta' \mathbf{R}'}, \tag{35}$$

where both continuity equation (26) and momentum equation (27) have been used and $\mathbf{I}$ is the unit tensor. Introducing the radiation-stress tensor $\mathbf{S}$ (i.e., Longuet-Higgins and Stewart[1]) for the depth-averaged velocity $\hat{\mathbf{u}}'$ and defining $\bar{\mathbf{R}}^S$ using Eq. (32) with $\mathbf{R}'$ instead of $\mathbf{u}'_\omega$, the advective term becomes

$$\overline{(\mathbf{u}'_\omega \cdot \nabla)\mathbf{u}'_\omega} = \frac{1}{h_s} \nabla \cdot \mathbf{S} + \frac{\partial \bar{\mathbf{u}}^S_\omega}{\partial t}$$
$$+ \mu^2 \frac{1}{h_s}[\nabla \cdot \mathbf{S}_2 - \overline{\nabla \cdot (h_s \mathbf{u}_2)\mathbf{u}'_\omega}] - \bar{\mathbf{R}}^S, \tag{36}$$

with

$$\mathbf{S} = h_s \overline{\hat{\mathbf{u}}'\hat{\mathbf{u}}'} + \mathbf{I}\frac{g}{2}\overline{\eta'^2}, \tag{37}$$

$$\mathbf{S}_2 \equiv \begin{pmatrix} 2\hat{u}u'_2 & u'_2\hat{v}' + v'_2\hat{u}' \\ u'_2\hat{v}' + v'_2\hat{u}' & 2\hat{v}v'_2 \end{pmatrix}. \tag{38}$$

From the vorticity equation for the waves, which can be easily derived taking the curl of the momentum equation (27), it follows that

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{u}'_\omega) = \nabla \times \mathbf{R}'. \tag{39}$$

This means that for $\nabla \times \mathbf{R}' \neq 0$ the waves are damped and vorticity is generated by the term $\mathbf{R}'$. In the momentum equation (34), we then find a *vortex-force* dispersive term $\mu^2 \mathbf{u}'_2 \times (\nabla \times \mathbf{u}'_\omega)$. Note that if $\nabla \times \mathbf{R}'$ is negligible at order $O(a)$, integrating Eq. (39) in time and assuming that there is no disturbance initially, it follows that $\nabla \times \mathbf{u}'_\omega \approx 0$ and the related dispersive contribution term can be neglected.

The momentum equation we get after these considerations, and using Eq. (36), reads at order $O(a^2)$,

$$\frac{\partial \bar{\mathbf{u}}^L_\omega}{\partial t} + g \nabla \bar{\eta} = \mathbf{F_s} + \mathbf{F_d} + \bar{\mathbf{R}}^L, \tag{40}$$

where

$$\mathbf{F_s} = -\frac{1}{h_s} \nabla \cdot \mathbf{S}, \tag{41}$$

$$\mathbf{F_d} = + \mu^2 \left[-\frac{1}{h_s} \nabla \cdot \mathbf{S}_2 + \frac{1}{h_s} \overline{\nabla \cdot (h_s \mathbf{u}_2') \mathbf{u}_\omega'} + \mathbf{F_{vf}} \right], \tag{42}$$

$$\mathbf{F_{vf}} = -\overline{\mathbf{u}_2' \times (\nabla \times \mathbf{u}_\omega')}. \tag{43}$$

$\bar{\mathbf{R}}^L$ is the dissipative body force acting directly on the Lagrangian-mean flow. The indirect effect of wave dissipation on the mean flow is implicit in Eqs. (41)–(43), and $\bar{\mathbf{R}}^L$ is usually small compared to the other forcing terms in Eq. (40) (i.e., Bühler[19]), and will be neglected in the following. Equation (40), together with Eq. (33), completely determines the Lagrangian-mean flow response (i.e., $\bar{\mathbf{u}}_\omega^L$, corresponding to the velocity $\mathbf{u}_\omega$) to the waves.

We observe that in Eq. (40), the radiation-stress term $\mathbf{F_s}$ is not the only forcing of the mean flow, as it happens in the shallow water framework (e.g., BJ01), but there are also dispersion contributions represented by $\mathbf{F_d}$. Note that the currents are expected to respond to various physical mechanisms such as wave dissipation, transience, and mean pressure changes due to waves, etc., and the radiation stress only describes some of these effects.

To exemplify the properties of the various contributions (40), we refer to a typical case of a rip-current flow (for details, see Kennedy *et al.*[11] and Terrile and Brocchini[13]), whose specific bathymetry is illustrated in the upper panel of Fig. 1. Using an updated version (i.e., Ch06) of FUNWAVE2D, a well-known Boussinesq model developed by the Center for Applied Coastal Research of the University of Delaware (for more details, see Wei *et al.*[17] and Chen *et al.*[8]), we numerically solve Eqs. (26) and (27) to quantify the role of the terms $\mathbf{F_s}$ and $\mathbf{F_d}$.

If the motion initially is irrotational, we may write

$$\mathbf{F_{vf}} = -\overline{\mathbf{u}_2 \times \int_0^t \nabla \times \mathbf{R}'}, \tag{44}$$

noting that there is often an initial period in which $\nabla \times \mathbf{R}'$ is negligible at order $O(a)$ (e.g., BJ01). On the other hand, the indirect effects of dissipation through the other wave-forcing terms are immediate. In the following, we therefore neglect $\mathbf{F_{vf}}$. In the lower four panels of Fig. 1, we compare the contributions of $\mathbf{F_d}$ (dashed line), $\mathbf{F_s}$ (dash-dotted line), and their sum (solid line) in the x and y directions at section A of the mentioned bathymetry, in the cross-shore direction (second-fourth panels), and in the long-shore direction (third-fifth panels). The second and third panels refer to a 0.07 m high wave field characterized by a wave period $T=1$ s (in this case, from our simulation we have $\mu=1.13$, in the middle of the rip channel at $x=10$ m), while for the fourth and fifth panels a shorter wave period, i.e., $T=0.9$ s (which corresponds to $\mu=1.31$ in the middle of the rip channel at $x=10$ m), has been used to get a more dispersive wave field. Since we used the same grid size for both simulations, the ratio wavelength-to-grid size is slightly larger for the longer wave, i.e., $L/\Delta x=12.5$ for $T=1.0$ s and $L/\Delta x=10.5$ for $T=0.9$ s. In turn, this results in the slightly smoother output of Fig. 1 for the shorter wave, this being discretized with fewer points. The comparison between $\mathbf{F_d}$ and $\mathbf{F_s}$ shows that the radiation stress term provides the largest contribution. In

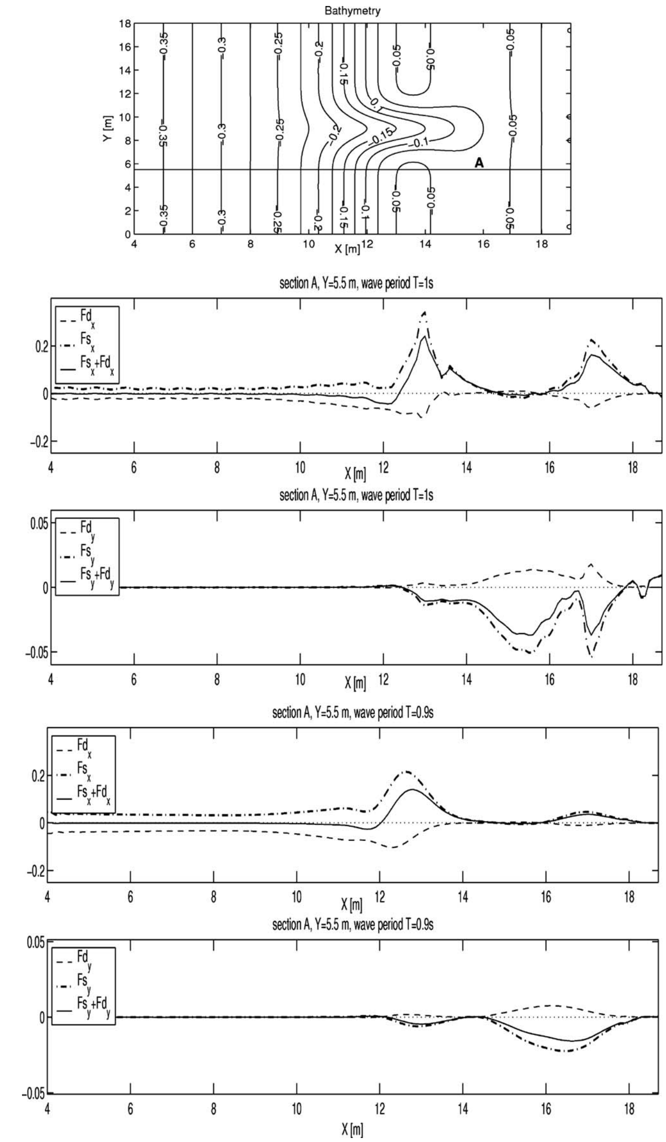

FIG. 1. (Top panel) Bathymetry of the chosen rip-channel test configuration (in meters). (Bottom four panels) Comparison between the $\mathbf{F_d}$ (dashed line), $\mathbf{F_s}$ (dash-dotted line) contributions to Eqs. (41) and (42) and their sum (solid line) at sections A (cross-shore and long-shore components, respectively, in the second-third and fourth-fifth panels) of the above bathymetry. Results are shown for the time $t=180$ s, for an offshore wave height $H_0=0.07$ m and wave period $T=1$ s (second and third panels) and $T=0.9$ s (fourth and fifth panels).

both cases, the x components of $\mathbf{F_s}$ and $\mathbf{F_d}$ are more relevant over the bar crest (i.e., 12 m$<x<$14 m) and in the swash zone (i.e., 16 m$<x<$18 m), while the y components assume higher values in correspondence of the trough (i.e., 14.5 m$<x<$17 m). The dispersive contributions all over the domain are of the same order of magnitude of the radiation stress components and, therefore, they can influence the forcing $\mathbf{F_s}+\mathbf{F_d}$. We observe that $\mathbf{F_d}$, in the above-mentioned zone, contributes to reducing the effect of $\mathbf{F_s}$. In particular, in the case in which dispersion is more important, because a shorter wave period has been used (i.e., fourth and fifth panels of Fig. 1), $\mathbf{F_d}$ becomes larger and closer in size and shape to $\mathbf{F_s}$ (i.e., fourth panel of Fig. 1, over the bar crest). These results provide an example of the more general finding for which $\mathbf{F_d}$ cannot be neglected with respect to $\mathbf{F_s}$ for accurate enough computations.

An important wave property is the so-called pseudomomentum per unit of mass $\mathbf{p}$ (see Andrews and McIntyre[14]). Neglecting the terms of order $O(\epsilon a^2)$, the pseudomomentum, associated with the velocity field $\mathbf{u}_\omega$, is given by

$$\mathbf{p} \approx \bar{\mathbf{u}}_\omega^S + \overline{\boldsymbol{\xi}' \times (\nabla \times \mathbf{u}'_\omega)}. \tag{45}$$

The corresponding pseudomomentum evolution equation can be easily derived multiplying Eq. (27) with η'/h_s, averaging and using the corresponding continuity equation (26). Invoking Eq. (45), we get (see BJ01)

$$\frac{\partial \mathbf{p}}{\partial t} + \frac{1}{h_s}\nabla \cdot \mathbf{S} - \frac{1}{2}\nabla \overline{|\mathbf{u}'_\omega|^2} = +\mu^2\left[-\frac{1}{h_s}\nabla \cdot \mathbf{S}_2 + \frac{1}{h_s}\overline{\nabla \cdot (h_s \mathbf{u}'_2)\mathbf{u}'_\omega}\right] + \boldsymbol{\mathcal{R}}, \tag{46}$$

with $\mathcal{R}_i = -\overline{\xi'_{j,i} R'_j}$. Note that $\boldsymbol{\mathcal{R}}$ is the same dissipative force described in Bühler[19] and, in the case of slowly varying mean quantities and a momentum-conserving dissipative body force, deriving from a stress tensor divergence, it turns out that $|\bar{R}_i^L|/|\mathcal{R}_i| = O(\epsilon)$.

To the present order of approximation, the effect of dispersion is additive; compare Eq. (3.37) of BJ01. Furthermore substituting Eq. (46) back into Eq. (40) and taking the curl gives

$$\frac{\partial}{\partial t}[\nabla \times (\bar{\mathbf{u}}_\omega^L - \mathbf{p})] = \nabla \times (-\boldsymbol{\mathcal{R}}) + \mu^2 \nabla \times \mathbf{F_{vf}}. \tag{47}$$

This equation gives a new result differing from the one obtained by BJ01 within the shallow-water framework because of the presence, on the r.h.s., of the last dispersive *vortex-force* term. In this case, i.e., in the presence of dissipation, the vorticity associated with $\bar{\mathbf{u}}_\omega^L - \mathbf{p}$ is no longer conserved as it is for shallow water flows, but it can be varyingly forced by the dispersion. Such conservation remains only if there is no dissipation and initial disturbance, i.e., $\nabla \times \mathbf{u}'_\omega \simeq 0 \rightarrow \mathbf{F_{vf}} \simeq 0$ everywhere.

The quasi-Eulerian mean flow is studied in the following assuming $\mathbf{p} \simeq \bar{\mathbf{u}}_\omega^S$, which holds at the third order, $O(a^3)$, for irrotational waves (for more details, see Andrews and McIntyre[14]). Considering depth-averaged velocity, as usually done in the shallow-water framework, we analyze the quasi-Eulerian velocity $\bar{\mathbf{u}}_\omega = \bar{\mathbf{u}}_\omega^L - \mathbf{p}$, splitting it in

$$\bar{\hat{\mathbf{u}}} + \mu^2 \bar{\mathbf{u}}_2 = \bar{\hat{\mathbf{u}}}^L - \mathbf{p}_\wedge + \mu^2(\bar{\mathbf{u}}_2^L - \mathbf{p}_2), \tag{48}$$

where it has been assumed that

$$\bar{\mathbf{u}}_\omega = \bar{\hat{\mathbf{u}}} + \mu^2 \bar{\mathbf{u}}_2, \tag{49}$$

$$\bar{\mathbf{u}}_\omega^L = \bar{\hat{\mathbf{u}}}^L + \mu^2 \bar{\mathbf{u}}_2^L, \tag{50}$$

$$\mathbf{p} = \mathbf{p}_\wedge + \mu^2 \mathbf{p}_2, \tag{51}$$

where $\mathbf{p}_\wedge$ is the depth-averaged part of the pseudomomentum, while $\mathbf{p}_2$ is its dispersive contribution.

Referring, again, to the rip-current configuration of Fig. 1 (upper panel), we observe a clear dominance of the cross-shore flows (in particular in the rip channel). Hence, inspection of the main flow properties (i.e., $\bar{\mathbf{u}}_\omega^L$, $\bar{\hat{\mathbf{u}}}^L$, $\mathbf{p}$, etc.) can be made by only referring to cross-shore components, the long-shore ones being very small and providing no evident indications. Figure 2 reports only the cross-shore components of the Lagrangian-mean velocities and pseudomomentum for the chosen test. Inspection of the figure reveals that, if reference is only made to the Lagrangian mean velocity fields, dispersive effects appear negligible. This is evident from the left column of Fig. 2. In particular, we observe that $(\bar{\hat{\mathbf{u}}}^L)_x$ (middle panel) is very similar to $(\bar{\mathbf{u}}_\omega^L)_x$ (upper panel), which means that $(\bar{\mathbf{u}}_2^L)_x$ is negligible (lower panel). The role of the dispersive effects becomes much clearer from inspection of the cross-shore components of the pseudomomentum $\mathbf{p}$ (shown in the right column of the same figure). Such quantity represents the actual wave momentum (i.e., the nonlinear forcing of the mean motion given by the waves) and differs from the classical "radiation stress" (which is a momentum flux), providing a more complete description of the wave-current interactions. From a comparison between the middle-right and the bottom-right panels of Fig. 2, both contributions, respectively, $(\mathbf{p}_\wedge)_x$ and $(\mathbf{p}_2)_x$, have the same intensity of order $(0.1-0.2)$ m/s, in correspondence of the rip channel. Therefore, it becomes evident that this wave forcing, $\mathbf{p}$ (upper-right panel), is influenced by the dispersion terms, being the corresponding component at order $O(\mu^2)$, $\mathbf{p}_2$, not negligible. Hence, we can rewrite Eq. (48) as

$$\bar{\hat{\mathbf{u}}} + \mu^2 \bar{\mathbf{u}}_2 \simeq \bar{\hat{\mathbf{u}}}^L - \mathbf{p}_\wedge - \mu^2 \mathbf{p}_2. \tag{52}$$

Considering that the depth-averaged flow is split as $\bar{\hat{\mathbf{u}}} = \bar{\hat{\mathbf{u}}}^L - \mathbf{p}_\wedge$, the dispersion, affecting the Eulerian flow through the term $\bar{\mathbf{u}}_2$, contributes mainly to influence the "wave forcing" $\mathbf{p}_2$, being the only two significant terms at order $O(\mu^2)$ in Eq. (52). In particular, such effects are more evident in regions where the horizontal velocity vertical shear is expected to be larger (i.e., in the rip channel) and, therefore, where the shallow-water approximation is no longer valid.

IV. DISCUSSION AND CONCLUSIONS

The results of the preceding section highlight the role of wave frequency dispersion on the response of the flow currents.

In the momentum equation, the radiation-stress term $\mathbf{F_s}$ is not the only forcing of the mean flow, as it happens in the shallow water framework (e.g., BJ01), but there are also dispersion contributions represented by $\mathbf{F_d}$. In particular, with reference to a typical rip-current configuration, we observe that such a radiation stress tensor term, $\mathbf{F_s}$, is the dominant forcing in a large part of the domain. The dispersive term $\mathbf{F_d}$ is relevant only over the bar crest and on the bar trough, contributing to changing the pseudomomentum $\mathbf{p}$ as stated by Eq. (46).

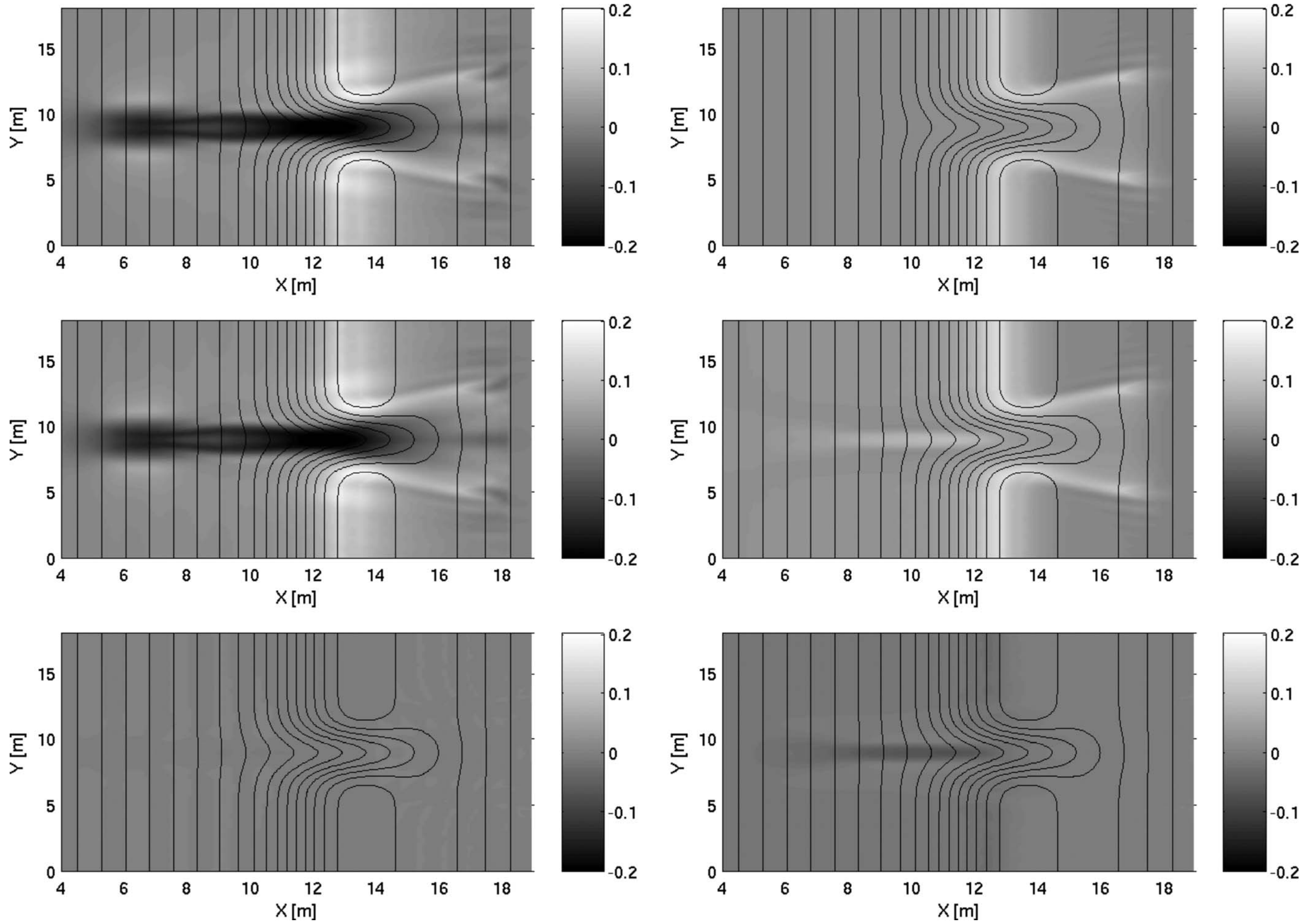

FIG. 2. Comparison between cross-shore components of the Lagrangian-mean velocities $\bar{\mathbf{u}}^L_\omega$ (upper-left panel), $\bar{\hat{\mathbf{u}}}^L$ (middle-left panel) and $\bar{\mathbf{u}}^L_2$ (bottom-left panel), and between the cross-shore component of the pseudomomentum $\mathbf{p}$ (upper-right panel), $\mathbf{p}_\wedge$ (middle-right panel), and $\mathbf{p}_2$ (bottom-right panel), at time $t=180$ s.

In contrast to dissipative forces, dispersion cannot contribute to any changes in the total energy of the system. Hence, while the Lagrangian mean motion is largely unaffected (i.e., Fig. 2), the dispersion is responsible for an increased transfer of momentum between the waves and the Eulerian mean flow $\bar{\mathbf{u}}^L_\omega - \mathbf{p}$.

Furthermore, dispersive effects on the vorticity dynamics, within the GLM-like framework, are analyzed. It has been made clear that proper description of the wave-current interaction and vorticity transport in the nearshore must account also for dispersive effects in such a system.

We find that dispersive effects are important in the description of the correct amount of the vorticity forcing. In particular, even in the absence of dissipation, the vorticity of the flow field $\bar{\mathbf{u}}^L_\omega - \mathbf{p}$ is no longer conserved.

Analyzing in details the quasi-Eulerian velocity field, we find that the dispersion contribution, affecting the depth-averaged Eulerian mean velocity field, mainly influences only the wave forcing (i.e., pseudomomentum) and not the Lagrangian mean flow field.

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APPENDIX: DERIVATION OF EQ. (33)

After averaging the continuity equation (16), keeping terms at order $O(a^2)$, we obtain

$$\frac{\partial \bar{\eta}}{\partial t} + \nabla \cdot [h_s(\bar{\mathbf{u}}_\omega - \mu^2 \bar{\mathbf{u}}_2)] + \nabla \cdot [\overline{\eta' \mathbf{u}'_\omega} - \mu^2 \overline{\eta' \mathbf{u}'_2}] = 0. \tag{A1}$$

Here, using Eq. (32), valid for the Stokes drift, and the corresponding similar expressions valid for any Stokes corrections, we find that

$$\overline{\eta' \mathbf{u}'_\omega} \approx h_s \bar{\mathbf{u}}^S_\omega \quad \text{and} \quad \overline{\eta' \mathbf{u}_2} \approx h_s \bar{\mathbf{u}}^S_2, \tag{A2}$$

therefore, since for any generic quantity ϕ, the relation $\bar{\phi}^L = \bar{\phi}^S + \bar{\phi}$ holds true, using Eqs. (A1) and (A2), we get Eq. (33) in the form

$$\frac{\partial \bar{\eta}}{\partial t} + \nabla \cdot [h_s(\bar{\mathbf{u}}^L_\omega - \mu^2 \bar{\mathbf{u}}^L_2)] = 0. \tag{A3}$$

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