BOUSSINESQ WAVES ON VERTICALLY SHEARED CURRENTS

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ABSTRACT

Boussinesq-type equations with second order dispersion are derived for an arbitrary distribution of vorticity. The governing equations have the velocity at an arbitrary depth as a dependent variable and terms involving vorticity are kept as integrals. Linear dispersive properties are shown to be accurate to the order of dispersion when compared to the exact solution. Current input conditions (from a large scale hydrodynamic model) and the scheme for the numerical model are obtained in order to later test model results against measurements from an experiment in a wave flume for the case of waves propagating against a vertically sheared current.

INTRODUCTION

Boussinesq-type wave models have proved to be efficient and accurate instruments for calculation of the transformation of nonlinear and dispersive waves in coastal waters where horizontal vorticity is considered null. There is actually no need to assume null horizontal vorticity though and Boussinesq equations provide an ideal framework for the treatment of the fully rotational problem. Recently, researchers have included the effect of horizontal vorticity in the governing equations in order to treat fully rotational problems, such as interaction of waves with vertically sheared currents (Rego and Neves 1997; Rego and Neves 2001) and wave breaking (Veeramony and Svendsen 2000). Incorporation of the interaction of nonlinear waves with depth varying currents within large scale coastal models is of great interest. The combined effects of waves and vertically sheared currents are important in coastal engineering problems ranging from determination of design parameters for coastal structures, estimation of flow kinematics near the bottom for sediment transport, to wind driven wave generation.

Two-dimensional Boussinesq-type equations with second order accurate dispersion are derived here for an arbitrary distribution of vorticity. The stream

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function is used to obtain the expression for the velocity. The depth-integrated governing equations have the velocity at an arbitrary depth as a dependent velocity variable and the terms involving vorticity are kept as integrals. Linear dispersive properties of the model are shown to be accurate to the order of dispersion of the model. Current input conditions (from a large scale hydrodynamic model) and the scheme for the numerical wave model are obtained in order to later test model results against measurements from an experiment in a wave flume for regular waves propagating against a vertically sheared current. It is assumed that the current is pre-existing and the vorticity is associated only to the current motion.

GOVERNING EQUATIONS

We use the definition of the velocity in the vertical plane in terms of derivatives of the scalar stream function ψ,

$$u = \psi_z \qquad w = -\psi_x \quad . \tag{1}$$

The vorticity is then given by

$$\xi = u_z - w_x \quad , \tag{2}$$

we obtain the equation for the stream function

$$\mu^2 \psi_{xx} + \psi_{zz} = \xi \quad . \tag{3}$$

The nondimensional variables are defined as follows:

$$x = \frac{x'}{l} \quad z = \frac{z'}{h_0} \quad t = \frac{\sqrt{gh_0}t'}{l} \quad u = \frac{h_0 u'}{a_0\sqrt{gh_0}} \quad w = \frac{h_0^2 w'}{a_0 l\sqrt{gh_0}} \tag{4}$$

$$\xi = \frac{h_0^2 \xi'}{a_0\sqrt{gh_0}} \quad \eta = \frac{\eta'}{a_0} \quad \psi = \frac{\psi'}{a_0\sqrt{gh_0}} \quad \mu = \frac{h_0}{l} \quad \epsilon = \frac{a_0}{h_0} \quad , \tag{5}$$

where the primes denote dimensional variables.

Integrating equation (3) twice from $-h$ to z, we obtain

$$\psi = (z+h)u_b + \int_{-h}^{z}\int_{-h}^{z} \xi dz^2 - \mu^2 \int_{-h}^{z}\int_{-h}^{z} \psi_{xx} dz^2 \quad , \tag{6}$$

where the following definitions have been made

$$u_b = \psi_z|_{z=-h} \qquad \psi|_{z=-h} = 0 \quad . \tag{7}$$

To the leading order, we have that

$$\psi_{xx} = u_b h_{xx} + 2h_x u_{bx} + h_x^2 \xi_b + (z+h)\left[u_{bxx} + h_x \xi_x|_{-h} + (h_x \xi_b)_x\right] + \int_{-h}^{z}\int_{-h}^{z} \xi_{xx} dz dz + O(\mu^2) \quad . \tag{8}$$

Note the extra terms, if compared to that obtained by Veeramony and Svendsen (2000), that depend on the vorticity at the bottom, which is not taken to be zero here.

The following expression for the stream function, up to second order dispersion, results:

$$\psi = (z+h)u_b + \int_{-h}^{z}\int_{-h}^{z} \xi dz^2 - \mu^2 \left\{ \frac{(z+h)^2}{2}(2h_x u_{bx} + u_b h_{xx} + h_x^2 \xi_b) \right.$$
$$\left. + \frac{(z+h)^3}{6}\left[u_{bxx} + h_x \xi_x|_{-h} + (h_x\xi_b)_x\right] + \int_{-h}^{z}\int_{-h}^{z}\int_{-h}^{z}\int_{-h}^{z} \xi_{xx} dz^4 \right\} + O(\mu^4) \quad . \quad (9)$$

We choose to work with the velocity at an arbitrary depth, $z_\beta = \beta h$, as the velocity variable as did Nwogu (1993), and find the following expressions for the horizontal and vertical velocities:

$$u = \tilde{u}_\beta + q_1 - \mu^2 \left[(Z_h - B_h)(2h_x\tilde{u}_{\beta x} + \tilde{u}_\beta h_{xx}) + \frac{1}{2}(Z_h^2 - B_h^2)\tilde{u}_{\beta xx} + q_2\right] \quad (10)$$

$$w(x,z,t) = -\mu^2[\tilde{u}_\beta h_x + Z_h \tilde{u}_{\beta x} + q_3] \quad , \quad (11)$$

where

$$\tilde{u}_\beta = u_\beta - \int_{-h}^{z_\beta} \xi dz \quad (12)$$

$$q_1(x,z,t) = \int_{-h}^{z} \xi dz \quad ; \quad q_3(x,z,t) = \frac{\partial}{\partial x}\int_{-h}^{z}\int_{-h}^{z} \xi dz^2 \quad (13)$$

$$q_2(x,z,t) = \int_{-h}^{z}\int_{-h}^{z}\int_{-h}^{z} \xi_{xx} dz^3 - \int_{-h}^{z_\beta}\int_{-h}^{z_\beta}\int_{-h}^{z_\beta} \xi_{xx} dz^3$$
$$+ \frac{1}{2}(Z_h^2 - B_h^2)h_x^2\xi_b + \frac{1}{6}(Z_h^3 - B_h^3)\left[h_x\xi_x|_{-h} + (h_x\xi_b)_x\right] \quad (14)$$

and the following notation has been introduced:

$$Z_h = z + h \quad ; \quad B_h = z_\beta + h \quad ; \quad H = \epsilon\eta + h \quad . \quad (15)$$

The resulting expression for pressure is:

$$p(x,z,t) = \eta - \frac{z}{\epsilon} - \mu^2 \left[(H - Z_h)\tilde{u}_{\beta t} h_x + \frac{1}{2}(H^2 - Z_h^2)\tilde{u}_{\beta tx} + \chi_3\right]$$
$$-\epsilon\mu^2 \left[(H - Z_h)(\tilde{u}_\beta^2 h_{xx} + \tilde{u}_\beta h_x \tilde{u}_{\beta x}) + \frac{1}{2}(H^2 - Z_h^2)(\tilde{u}_\beta\tilde{u}_{\beta xx} - \tilde{u}_{\beta x}^2) + \chi_4\right] + O(\mu^4) \quad . \quad (16)$$

The mass and momentum equations are given by:

$$\eta_t + \frac{\partial}{\partial x}\left\{ H\tilde{u}_\beta + \chi_1 - \mu^2 \left[\left(\frac{H^2}{2} - HB_h\right)(2h_x\tilde{u}_{\beta x} + \tilde{u}_\beta h_{xx}) \right.\right.$$
$$\left.\left. + \left(\frac{H^3}{6} - \frac{HB_h^2}{2}\right)\tilde{u}_{\beta xx} + \chi_2 \right]\right\} = O(\mu^4) \quad (17)$$

$$\tilde{u}_{\beta t} + \eta_x + \chi_5 + \epsilon(\tilde{u}_\beta \tilde{u}_{\beta x} + \chi_6) + \mu^2(V_1 + \chi_7) + \epsilon\mu^2(V_2 + \chi_8) = O(\mu^4) \quad , \qquad (18)$$

where

$$V_1 = \frac{z_\beta^2}{2}\tilde{u}_{\beta txx} + z_\beta(h\tilde{u}_{\beta t})_{xx} - \left[\frac{(\epsilon\eta)^2}{2}\tilde{u}_{\beta tx} + \epsilon\eta(h\tilde{u}_{\beta t})_x\right]_x \qquad (19)$$

$$V_2 = \left\{\frac{1}{2}[(h\tilde{u}_\beta)_x + \epsilon\eta\tilde{u}_{\beta x}]^2 - (z_\beta - \epsilon\eta)\tilde{u}_\beta(h\tilde{u}_\beta)_{xx} - \frac{1}{2}[z_\beta^2 - (\epsilon\eta)^2]\tilde{u}_\beta\tilde{u}_{\beta xx}\right\}_x \quad . \qquad (20)$$

The terms that depend on the vertical profile of vorticity are given by:

$$\chi_1 = \int_{-h}^{\epsilon\eta} q_1 dz \qquad \chi_2 = \int_{-h}^{\epsilon\eta} q_2 dz \qquad \chi_3 = \frac{\partial}{\partial t}\int_z^{\epsilon\eta} q_3 dz \qquad (21)$$

$$\chi_4 = \frac{\partial}{\partial x}\int_z^{\epsilon\eta} [(\tilde{u}_\beta h_x + Z_h\tilde{u}_{\beta x})q_1 + (\tilde{u}_\beta + q_1)q_3]dz + [2(\tilde{u}_\beta h_x + Z_h\tilde{u}_{\beta x}) + q_3]q_3 - (\tilde{u}_\beta h_x + H\tilde{u}_{\beta x})\chi_{1x} \qquad (22)$$

$$\chi_5 = \frac{1}{H}\frac{\partial\chi_1}{\partial t}; \qquad \chi_6 = \frac{1}{H}\left[\frac{\partial}{\partial x}\int_{-h}^{\epsilon\eta}(2\tilde{u}_\beta + q_1)q_1 dz - \tilde{u}_\beta\chi_{1x}\right] \qquad (23)$$

$$\chi_7 = \frac{1}{H}\left(\chi_3|_{-h}h_x - \frac{\partial\chi_2}{\partial t} - \frac{\partial}{\partial x}\int_{-h}^{\epsilon\eta}\chi_3 dz\right) \qquad (24)$$

$$\chi_8 = \frac{1}{H}\Big\{\chi_4|_{-h}h_x + \left[(H - B_h)(2\tilde{u}_{\beta x}h_x + \tilde{u}_\beta h_{xx}) + \frac{1}{2}(H^2 - B_h^2)\tilde{u}_{\beta xx}\right]\chi_{1x} + \tilde{u}_\beta\chi_{2x} - \frac{\partial}{\partial x}\int_{-h}^{\epsilon\eta}\{\chi_4 + 2(\tilde{u}_\beta + q_1)q_2 + (Z_h - B_h)[(4\tilde{u}_{\beta x}h_x + 2\tilde{u}_\beta h_{xx}) + (Z_h + B_h)\tilde{u}_{\beta xx}]\,q_1\}\,dz\Big\} \quad . \qquad (25)$$

LINEAR DISPERSIVE PROPERTIES

From here on, the variables will be dimensional and the primes will be dropped. The velocity, $\tilde{u}_\beta$, is decomposed into current, $\tilde{u}_\beta^c = u_\beta^c + \int_{-h}^{z_\beta}\xi dz$, and wave, $\tilde{u}_\beta^w = u_\beta^w$, components, assuming that the vorticity is only associated to the current motion. The linearized governing equations for a flat bottom, steady and spatially uniform current with arbitrary over depth vorticity read

$$\eta_t + (\tilde{u}_\beta^c + h\gamma_0)\eta_x + h\tilde{u}_{\beta x}^w + (\alpha + 1/3)\,h^3\tilde{u}_{\beta xxx}^w = 0 \qquad (26)$$

$$\tilde{u}_{\beta t}^w + \eta_x + (\tilde{u}_\beta^c + \gamma_1 - \gamma_0 h)\tilde{u}_{\beta x}^w + \alpha h^2\tilde{u}_{\beta txx}^w + \left[\alpha h^2\tilde{u}_\beta^c - \gamma_2 - \gamma_0\,(\alpha + 1/3)\,h^3\right]\tilde{u}_{\beta xxx}^w = 0 \quad , \qquad (27)$$

where three terms depending on the vertical distribution of vorticity appear:

$$\gamma_0 = \frac{1}{h}\int_{-h}^{0} \xi dz \qquad\qquad \gamma_1 = \frac{2}{h}\int_{-h}^{0}\int_{-h}^{z} \xi dz^2 \tag{28}$$

$$\gamma_2 = \frac{1}{h}\left\{\int_{-h}^{0}\left[\left(Z_h^2 - B_h^2\right)\int_{-h}^{z}\xi dz\right]dz + \int_{-h}^{0}\int_{z}^{0}\left(Z_h\int_{-h}^{z}\xi dz\right)dz^2\right\} \quad . \tag{29}$$

In the case of a linear and plane wave, it results in the following dispersion relation:

$$\hat{\omega}^2[1-\alpha(kh)^2] = (gk - \hat{\omega}\gamma_0)kh[1-(\alpha+1/3)\,(kh)^2] + \hat{\omega}k(\gamma_1 + k^2\gamma_2) - \hat{\omega}kh\gamma_0[1-\alpha(kh)^2] \quad , \tag{30}$$

where the intrinsic frequency is given by:

$$\hat{\omega} = \omega - ku_s^c \tag{31}$$

and $u_s^c = \tilde{u}_\beta^c + \gamma_0 h$ is the current velocity near the surface.

Figure 1 shows the normalized model phase velocity, relative to the exact phase velocity obtained from the Raleigh equation (see Appendix 1), which shows the same order of error of the extended second order Boussinesq models without horizontal vorticity (Nwogu 1993).

NUMERICAL MODEL

The numerical model is based on an implementation of the scheme described in Wei et al. (1995), only with the new matrices E_r and F_r containing the horizontal vorticity terms:

$$\eta_t = E(\eta, \tilde{u}_\beta) + E_2(\eta, \tilde{u}_\beta) + E_r(\eta, \xi) \tag{32}$$

$$U_t(\tilde{u}_\beta) = F(\eta, \tilde{u}_\beta) + F_2(\eta, \tilde{u}_\beta, \tilde{u}_{\beta t}) + F_r(\eta, \tilde{u}_\beta, \xi) \quad , \tag{33}$$

where

$$E_r = (-\chi_1 + \chi_2)_x \qquad F_r = -(\chi_5 + \chi_6 + \chi_7 + \chi_8) \quad . \tag{34}$$

The time derivatives in F_r are treated the same manner as those in F_2. The vertical distribution of vorticity due to the pre-existing current is known and approximated by Chebyshev polynomials in E_r and F_r. All intermediate terms involving vorticity are calculated using special properties of the Chebyshev polynomials which allow for the use of FFT.

The incident boundary condition is obtained from the linearized mass equation, which reads:

$$\eta_t + (\tilde{u}_\beta^c + h\gamma_0)\eta_x + h\tilde{u}_{\beta x}^w + (\alpha + 1/3)\,h^3\tilde{u}_{\beta xxx}^w = 0 \quad . \tag{35}$$

The wave component of the velocity is obtained:

$$\tilde{u}_\beta^w = \frac{\hat{\omega}}{kh[1-(\alpha+1/3)\,(kh)^2]}\eta \quad , \tag{36}$$

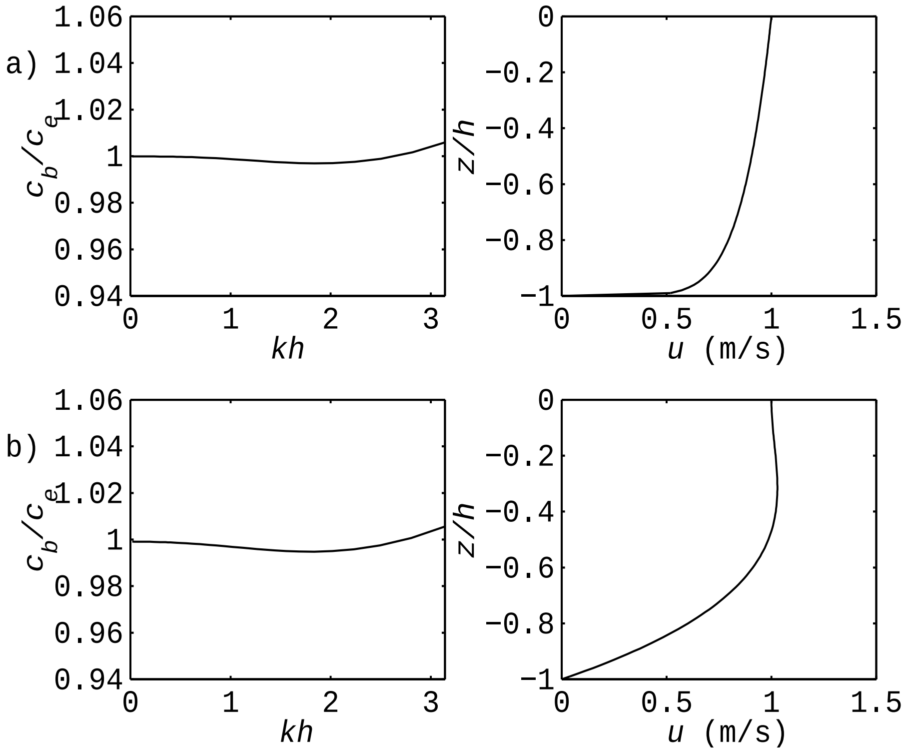

FIG. 1. Normalized phase velocity (model phase velocity, c_b, divided by the exact phase velocity calculated from the Raleigh equation, c_e) and associated current profiles: a) 1/7 power law and b) cubic polynomial.

where the intrinsic frequency is given by (31), and results in the total velocity:

$$\tilde{u}_\beta = \tilde{u}^c_\beta + \frac{\hat{\omega}}{kh[1-(\alpha+1/3)\,(kh)^2]}\eta \quad . \tag{37}$$

When reflection at the incident boundary is significant, a non-reflective boundary condition is implemented:

$$\eta_t - c_R \eta_x = \left(1 + \frac{c_R}{c_I}\right)\eta_{It} \quad , \tag{38}$$

where I and R stand for incident and reflected wave, respectively.

When absorbing waves and transmitting currents, the following scheme is used for both the velocity and surface elevation variables

$$u_i = u_i^{ref} + (u_i^{ref} - u_i)/C_s(i) \tag{39}$$

$$C_s\,(i) = \alpha_s^{\gamma_s nx-i} \quad \text{for} \quad i = nx-L, ..nx \quad , \tag{40}$$

as in Chen (1997), where ref indicates values due to the pre-existing current.

TEST CASE

The laboratory experiment that will be used to test the model consisted of regular waves propagating against a strongly sheared current over a trapezoidal bar as depicted in Figure 2. Surface elevation and velocity time series along 9 to 14 depths were measured at 11 different locations.

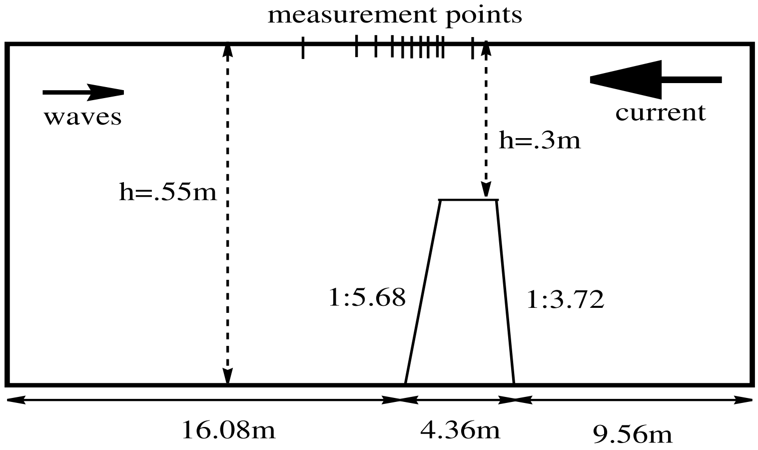

FIG. 2. Experimental setup.

Current modeling

In order to obtain densely spaced current data to be used as input to the wave model, the current in the wave tank was simulated using the Princeton Ocean Model (POM) (Blumberg and Mellor 1973). POM is a quasi-three-dimensional hydrodynamic model, which assumes hydrostatic pressure and includes a turbulence closure model, therefore permitting the representation of the bottom boundary layer. The results shown in Figure 3 were obtained using a 2DV version of POM, inflow and outflow boundaries were posed for both external (vertically averaged equations) and internal (solution of vertical structure) modes. The vertical dimension was discretized using 21 grid points with varying (log) spacing near the bottom, horizontal spacing of .05 m was used with a time step of .1953 s for both the internal and external modes. Figure 3 shows the importance of the shear due to the bottom boundary layer that is reproduced by POM. The discrepancies between the measured current profile and POM data are believed to be resulting from three dimensional effects in the wave tank, and may be further investigated elsewhere.

Wave modeling

Current research efforts are concentrated on the wave modelling and comparison of model results with experimental measurements and will be published in a forthcoming publication.

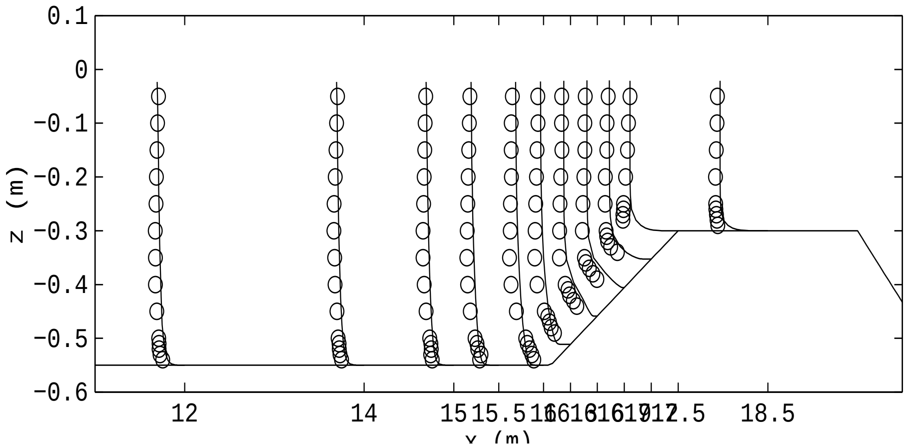

FIG. 3. Measured (o) and POM output (–) values for horizontal current velocity.

CONCLUSION

Boussinesq-type equations of second order dispersion were obtained for an arbitrary distribution of vorticity. Linear dispersive properties of the model were shown to be accurate to the order of dispersion of the model when compared to the exact solution of the Raleigh equation for idealized current profiles. Current input conditions for the wave model and the scheme for the numerical model presented here will be used to test model results against measurements from an experiment in a wave flume for the case of waves propagating against a vertically sheared current. and over a trapezoidal bar. Further investigation of the nonlinear interactions between waves and sheared currents should include changes in current profile (since up until now, the model has only been used to calculate the transformation of waves on a pre-existing current).

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APPENDIX I. EXACT SOLUTION TO THE RALEIGH EQUATION

For a stream function given by $\psi = f(z)e^{ik(x-ct)}$, we have that the vertical distribution function, f, is given by the Raleigh equation:

$$(c-U)(\frac{d^2f}{dz^2} - k^2 f) + \frac{d^2U}{dz^2} f = 0 \qquad \text{for} \quad -h < z < 0 \quad , \tag{41}$$

TABLE 1. Expressions for γ and c for idealized current profiles.

$U(z)$	$\gamma^2(z')$	$c^2/\sqrt{gh}$
$U_s(1+z')^{1/7}$	$k^2h^2 + \frac{6}{49}\frac{\delta(1+z')^{-13/7}}{[1-\delta(1+z')^{1/7}]}$	$\frac{q(0)}{(\delta-1)[\delta-1-\frac{1}{7}\delta q(0)]}$
$U_s(1+\sigma_1 z'+\sigma_2 z'^2+\sigma_3 z'^3)$	$k^2h^2 + \frac{\delta(2\sigma_2+6\sigma_3 z')}{\delta[1+\sigma_1 z'+\sigma_2 z'^2+\sigma_3 z'^3]-1}$	$\frac{q(0)}{(\delta-1)[\delta-1-\sigma_1\delta q(0)]}$

with boundary conditions

$$(U-c)^2\frac{df}{dz} = [g + \frac{dU}{dz}(U-c)]f \qquad \text{at} \quad z=0 \tag{42}$$

$$f = 0 \qquad \text{at} \quad z=-h \quad . \tag{43}$$

Defining:

$$W(z) = U(z) - c \qquad z' = \frac{z}{h} \qquad q(z') = \frac{f(z)}{h\frac{df(z)}{dz}} \quad , \tag{44}$$

the Riccati equation is obtained:

$$\frac{dq}{dz'} = 1 - \gamma^2(z')q^2 \quad , \quad \text{where} \qquad \gamma^2(z') = k^2h^2 + h^2\frac{d^2W}{dz^2}\frac{1}{W} \tag{45}$$

and now the free surface boundary condition is given by:

$$q(0) = \frac{[F(0)]^2}{1+F(0)\frac{dF}{dz}(0)} \text{ where } F(0) = \frac{W(0)}{\sqrt{gh}} \text{ and } \frac{dF}{dz}(0) = \frac{dW}{dz}(0)\sqrt{\frac{h}{g}} \; . \tag{46}$$

Table 1 presents expressions for γ^2 and c (as a function of q) for idealized current profiles, $U(z)$, for the cases of a 1/7 power law type profile obtained by Fenton (1973) and for a polynomial profile obtained here, where $\delta = \frac{U_s}{C}$.

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