Chapter 5

Parabolic Modelling of Water Waves

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Abstract

The propagation of linear water waves over a three-dimensional ocean is modelled using the mild-slope equation. Various parabolic wave models are described that approximate the governing elliptic partial differential equation, and so are very convenient for computing wave propagation over large distances. Several aspects are discussed: computation of the reflected wavefield, the construction of good lateral boundary conditions (also known as 'non-reflecting boundary conditions'), the modelling of porous regions, and the application of conformal mapping to simplify the geometry of the computational domain. Parallel work using the Boussinesq equations for weakly nonlinear, weakly dispersive long waves is then reviewed.

1 Introduction

The propagation of water waves over a three-dimensional ocean is governed by Laplace's equation, which is an elliptic partial differential equation, in a variable domain. As the numerical solution of such equations

over large areas is computationally intensive, parabolic approximations have been developed: they reduce the number of spatial dimensions to two and also remove some of the boundary condition requirements. These approximations are the subject of this paper.

Parabolic approximations are appropriate when the waves propagate mainly in one direction, taken to be the x-direction, and lead to parabolic partial differential equations; these can be solved numerically by marching in x.

Parabolic approximations were first used by Leontovich and Fock in the 1940's to obtain analytical approximations for certain high-frequency diffraction problems. See Nussenzveig [1, §7.3] for a recent discussion. Tappert [2, Appendix A] gives a brief historical survey.

Nowadays, the main virtue of parabolic approximations is the ease with which such partial differential equations can be solved numerically. This virtue was recognised first in seismology; see Claerbout's book [3]. However, it is in the field of underwater acoustics that most developments have occurred. Here, the basic problem is to calculate the propagation of sound waves over vast distances through a compressible ocean. For reviews, see Tappert [2] or Ames & Lee [4] as well as the text by Jensen *et al.* [5]. For parabolic approximations in elastodynamics, see McCoy [6, 7].

Parabolic approximations were first used in the context of three-dimensional water waves by Liu & Mei [8] and Radder [9]. Mei & Tuck [10] developed a linear parabolic model of waves propagating past slender bodies. Kirby & Dalrymple [11] and Liu & Tsay [12] showed how weak nonlinear effects (corresponding to intermediate depth, dispersive Stokes waves) can be included. The effects of current and variable bathymetry were explored by Booij [13], Liu [14], and Kirby [15]. Stochastic variations in the ocean depth were considered by Reeve [16]. Other developments are described in [17] and [18], and in other papers cited below.

In this paper, we start from the mild-slope equation, recast as a Helmholtz equation with a spatially-varying wavenumber. We then describe various methods for deriving parabolic approximations to this equation. Three further topics are discussed, driven mainly by computational considerations. First, it is most convenient to solve the chosen parabolic equation on a rectangular grid in the (x, y)-plane, marching in x. This leads to a study of appropriate lateral boundary conditions (on lines $y =$ constant). We describe a 'perfect boundary condition' that is transparent to any waves leaving the computational domain. (This is a non-standard example of so-called 'non-reflecting boundary conditions'.) Second, we consider the effect of porous regions (such as rubble breakwaters) adjacent to the water; a perfect boundary condition is developed for such situations. Third, we consider the application of conformal mappings, so as to map complicated geometries onto rectangular computational domains.

When waves propagate into shallow water, dispersive effects become small and the separate frequency components of a wave train become coupled through weakly detuned three-wave interactions. This physical system is usually modelled at present using the Boussinesq equations, and a development of parabolic approximations to these equations has proceeded that mirrors the development for the mild-slope equation in many ways. We conclude by reviewing the developments in this area.

2 Governing equations

Choose Cartesian coordinates $Oxyz$, so that $z = 0$ is the undisturbed free surface, with the z-axis pointing upwards. The bottom is at

$$z = -h(x, y) = -h(\boldsymbol{x}),$$

where we use $\boldsymbol{x} = (x, y)$ for the horizontal coordinates. We assume that the water is incompressible and inviscid, and that the motion is irrotational; thus, a velocity potential

$$\Phi(x, y, z, t) = \mathrm{Re}\left\{\phi(x, y, z)\,\mathrm{e}^{-\mathrm{i}\omega t}\right\}$$

exists, where ω is the circular frequency. The potential ϕ satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = 0$$

in the water and

$$\frac{\partial \phi}{\partial n} = 0 \qquad \text{on } z = -h(\boldsymbol{x}),$$

where $\partial/\partial n$ denotes normal differentiation.

At the free surface, $z = \eta(x, y, t)$, there are two nonlinear boundary conditions [19],

$$\frac{\partial \eta}{\partial t} - \frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial \eta}{\partial y} = 0$$

and

$$g\eta + \frac{\partial \Phi}{\partial t} + \frac{1}{2}\,|\mathrm{grad}\,\Phi|^2 = 0,$$

where g is the acceleration due to gravity. A scaling argument allows linearization based on the wave steepness, ka, where k is a wavenumber and a is a characteristic amplitude. Thus the linear velocity potential satisfies the linearized condition

$$\omega^2\phi = g\frac{\partial \phi}{\partial z} \qquad \text{on } z = 0. \tag{1}$$

The surface elevation is given by

$$\eta(\boldsymbol{x}, t) = \mathrm{Re}\left\{\zeta(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega t}\right\} = -\frac{1}{g}\frac{\partial \Phi}{\partial t}$$

on $z = 0$, whence

$$\zeta(\boldsymbol{x}) = \frac{\mathrm{i}\omega}{g}\,\phi(x, y, 0).$$

2.1 Constant depth

If the water has constant depth h, we can separate out the dependence on z. Thus, we can write the linear potential as

$$\phi(x, y, z) = -\frac{\mathrm{i}g}{\omega}\,\zeta(\boldsymbol{x})\,\frac{\cosh k(h+z)}{\cosh kh}. \tag{2}$$

The two-dimensional complex-valued function ζ satisfies the Helmholtz equation

$$(\nabla^2 + k^2)\zeta = 0, \tag{3}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the two-dimensional Laplacian in the horizontal plane. Given ω, the wavenumber k is defined to be the unique positive real root of

$$\omega^2 = gk\tanh kh; \tag{4}$$

this dispersion relation ensures that the free-surface condition (1) and the rigid-bottom condition,

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{on } z = -h,$$

are both satisfied.

A typical wave-like solution of (3) is

$$\zeta(\boldsymbol{x}) = \mathrm{e}^{\mathrm{i}(\ell x + my)},$$

where $\ell = k\cos\theta$ and $m = k\sin\theta$. This corresponds to a plane wave of unit amplitude propagating at an angle θ to the x-axis. The phase speed is

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh kh}, \tag{5}$$

whereas the group velocity is

$$c_{\mathrm{g}} = \frac{d\omega}{dk} = \tfrac{1}{2}c\left\{1 + \frac{2kh}{\sinh 2kh}\right\}. \tag{6}$$

2.2 Variable depth: the mild-slope equation

If the depth h varies with $\boldsymbol{x}$, it is not possible, in general, to reduce the three-dimensional Laplace equation to a two-dimensional partial differential equation in the horizontal plane. However, an approximate reduction can be made, resulting in

$$\frac{\partial}{\partial x}\left(p\frac{\partial \zeta}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial \zeta}{\partial y}\right) + k^2 p\zeta = 0. \tag{7}$$

Here, the local wavenumber $k(\boldsymbol{x})$ is determined from the dispersion relation (4), using the water depth at $\boldsymbol{x}$, and then $p(\boldsymbol{x})$ is defined by

$$p(\boldsymbol{x}) = c(\boldsymbol{x})\,c_{\mathrm{g}}(\boldsymbol{x}),$$

using (5) and (6).

Equation (7) is called the *mild-slope equation*. It was derived by Berkhoff [20], using regular perturbation expansions and an integration over the water depth; see also [21] and [19, §3.5]. It has the same form as the equation governing the acoustic pressure (ζ) in a compressible fluid with variable density ($1/p$).

The mild-slope equation is exact for deep water and for water of constant finite depth (when it reduces to (3)). For shallow water ($kh \to 0$), it reduces to the shallow-water equation (formally, replace $p(\boldsymbol{x})$ by $h(\boldsymbol{x})$ in (7)). Finally, it is known that the mild-slope equation is valid for intermediate depths, provided $h(\boldsymbol{x})$ does not change too rapidly over a wavelength [22].

For information on the numerical treatment of the mild-slope equation, see Berkhoff *et al.* [23], Ebersole [24], Li & Anastasiou [25], Martin & Dalrymple [26] and other papers in this book.

Make the standard substitution [27]

$$\psi(\boldsymbol{x}) = \sqrt{p(\boldsymbol{x})}\,\zeta(\boldsymbol{x})$$

in (7); the result is

$$(\nabla^2 + K^2)\psi = 0,$$

where

$$K^2 = k^2 - \frac{\nabla^2\sqrt{p}}{\sqrt{p}}. \tag{8}$$

Thus, we see that the mild-slope equation can be written as a two-dimensional Helmholtz equation with a spatially-varying wavenumber, $K(\boldsymbol{x})$. This reduction was used by Radder [9].

Various modifications of the mild-slope equation have been proposed. Thus, Kirby [28] derived a modified equation for rippled beds, using Green's Identity. Massel [29] developed an equation for a rapidly varying bathymetry, using an eigenfunction expansion including the evanescent modes. His equation reduces to Berkhoff's for mild slopes.

Bottom friction, seaweed, wave breaking, porous bottoms and mud bottoms can all contribute to a damping of wave energy. Dalrymple *et al.* [30], following Booij [13], show how to incorporate these effects via a complex dissipation function w which modifies the mild-slope equation (7) by adding a term

$$-\mathrm{i}\omega w\zeta$$

to the left-hand side; equivalently, add the term

$$-\frac{\mathrm{i}kw}{c_\mathrm{g}}.$$

to the right-hand side of (8). The wave-breaking model of Dally *et al.* [31] was incorporated by Kirby & Dalrymple [32].

3 Parabolic models for the Laplace and Helmholtz equations

We have seen that our water-wave problem can be reduced to the Helmholtz equation, which we rewrite here as

$$\nabla^2\psi + [k(\boldsymbol{x})]^2\psi = 0. \tag{9}$$

Where appropriate, we use k_0 to denote a *constant* wavenumber, giving

$$\nabla^2\psi + k_0^2\psi = 0. \tag{10}$$

In this section, we shall give several derivations of several parabolic approximations to the Helmholtz equation.

3.1 Heuristic derivations

Look for a solution of (9) in the form

$$\psi(\boldsymbol{x}) = u(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}k_0 x}, \tag{11}$$

where the constant k_0 may be thought of as an average of $k(\boldsymbol{x})$. The equation for u is found to be

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\mathrm{i}k_0\frac{\partial u}{\partial x} + (k^2 - k_0^2)u = 0.$$

Next, we discard the term $\partial^2 u/\partial x^2$. This may be justified by supposing that $u(x, y)$ is a slowly-varying function of x, whence

$$\left|\frac{\partial u}{\partial x}\right| \ll |k_0 u|. \tag{12}$$

Physically, this is reasonable if most of the variation of ψ with x is given by the exponential in (11). The resulting equation is

$$\frac{\partial u}{\partial x} = \frac{\mathrm{i}}{2k_0}\left\{(k^2 - k_0^2)u + \frac{\partial^2 u}{\partial y^2}\right\}, \tag{13}$$

or, reverting to ψ (using $u = \psi\,\mathrm{e}^{-\mathrm{i}k_0 x}$),

$$\frac{\partial \psi}{\partial x} = \frac{\mathrm{i}}{2k_0}\left\{(k^2 + k_0^2)\psi + \frac{\partial^2 \psi}{\partial y^2}\right\}. \tag{14}$$

In particular, if k is constant, $k = k_0$, equations (13) and (14) reduce to

$$2\mathrm{i}k_0\,\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{15}$$

and

$$\frac{\partial \psi}{\partial x} = \mathrm{i}k_0\psi + \frac{\mathrm{i}}{2k_0}\frac{\partial^2 \psi}{\partial y^2}, \tag{16}$$

respectively. We call these the *simple parabolic equations.*

More generally, we can consider the WKB-type representation,

$$\psi(\boldsymbol{x}) = u(\boldsymbol{x}) \exp\left\{ \mathrm{i} \int^x k_1(x')\, dx' \right\},$$

where k_1 is a given function of one variable. Proceeding as before, the equation for u is found to be

$$2\mathrm{i}k_1 \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial k_1}{\partial x} u + (k^2 - k_1^2)u + \frac{\partial^2 u}{\partial y^2} = 0, \tag{17}$$

after discarding the term $\partial^2 u/\partial x^2$. This equation reduces to (13) when $k_1(x) = k_0$.

3.2 Wave splitting

The presentation here is based mainly on McDaniel's papers [33, 34]. We start by writing the Helmholtz equation (9) as a first-order system,

$$\frac{\partial}{\partial x}\begin{pmatrix} \psi \\ \partial\psi/\partial x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\mathcal{P}^2 & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \partial\psi/\partial x \end{pmatrix}, \tag{18}$$

where $\mathcal{P}^2$ is an operator, defined by

$$\mathcal{P}^2 = k^2 + \frac{\partial^2}{\partial y^2}.$$

Next, we aim to split ψ into the sum of two functions, corresponding to waves travelling in opposite directions; we are interested in those waves travelling in the direction of x increasing. So, we introduce the 2×2 *splitting matrix* T,

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

and then define the forward (ψ^+) and backward (ψ^-) fields by

$$\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = T \begin{pmatrix} \psi \\ \partial\psi/\partial x \end{pmatrix}. \tag{19}$$

The matrix T can be chosen in many ways. We want to have the decomposition

$$\psi = \psi^+ + \psi^-,$$

and this implies that

$$\alpha + \gamma = 1 \qquad \text{and} \qquad \beta + \delta = 0.$$

In addition, for T to be invertible, we require that $\beta \neq 0$.

Combining equations (18) and (19), we obtain

$$\frac{\partial}{\partial x}\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = \left[\frac{\partial T}{\partial x} + T\begin{pmatrix} 0 & 1 \\ -\mathcal{P}^2 & 0 \end{pmatrix}\right] T^{-1}\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}.$$

Written explicitly, this pair of equations becomes

$$\frac{\partial \psi^+}{\partial x} = \mathcal{A}^+\psi^+ + \mathcal{A}^-\psi^-, \tag{20}$$

$$\frac{\partial \psi^-}{\partial x} = \mathcal{B}^+\psi^+ + \mathcal{B}^-\psi^-, \tag{21}$$

$$\mathcal{A}^+ = \frac{\partial \alpha}{\partial x} - \beta\mathcal{P}^2 + \frac{1-\alpha}{\beta}\left(\frac{\partial \beta}{\partial x} + \alpha\right),$$

$$\mathcal{A}^- = \frac{\partial \alpha}{\partial x} - \beta\mathcal{P}^2 - \frac{\alpha}{\beta}\left(\frac{\partial \beta}{\partial x} + \alpha\right)$$

$$= \beta\left\{\frac{\partial}{\partial x}\left(\frac{\alpha}{\beta}\right) - \left(\frac{\alpha}{\beta}\right)^2 - k^2 - \frac{\partial^2}{\partial y^2}\right\},$$

$$\mathcal{B}^+ = -\frac{\partial \alpha}{\partial x} + \beta\mathcal{P}^2 - \frac{1-\alpha}{\beta}\left(\frac{\partial \beta}{\partial x} - (1-\alpha)\right),$$

$$\mathcal{B}^- = -\frac{\partial \alpha}{\partial x} + \beta\mathcal{P}^2 + \frac{\alpha}{\beta}\left(\frac{\partial \beta}{\partial x} + (1-\alpha)\right).$$

So far, we have not made any approximations. We now do two things: we make a choice for T and we discard the reflected wave ψ^- and the term $\mathcal{A}^-\psi^-$ from (20). Different choices of T lead to different parabolic approximations.

When $k = k_0$, a constant, we would like the equations for ψ^+ and ψ^- to decouple. In particular, we should admit the plane-wave solutions,

$$\psi^+ = \mathrm{e}^{\mathrm{i}k_0 x} \qquad \text{and} \qquad \psi^- = \mathrm{e}^{-\mathrm{i}k_0 x}.$$

We can arrange for this by choosing $(\alpha/\beta)^2 + k^2 = 0$, which leads to two choices for the splitting matrix, namely

$$T_0 = \tfrac{1}{2}\begin{pmatrix} 1 & -\mathrm{i}/k_0 \\ 1 & \mathrm{i}/k_0 \end{pmatrix} \qquad \text{and} \qquad T_1 = \tfrac{1}{2}\begin{pmatrix} 1 & -\mathrm{i}/k \\ 1 & \mathrm{i}/k \end{pmatrix}.$$

If we use T_0 (so that $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}\mathrm{i}/k_0$), we find that

$$\mathcal{A}^- = \frac{\mathrm{i}}{2k_0}\frac{\partial^2}{\partial y^2},$$

so that the equations for ψ^+ and ψ^- do not decouple. Nevertheless, if we discard the term involving $\mathcal{A}^-$, we see that (20) reduces to (14).

Similarly, if we use T_1 (so that $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}\mathrm{i}/k$), we find that

$$\mathcal{A}^- = \frac{1}{2k}\left(\frac{\partial k}{\partial x} + \mathrm{i}\frac{\partial^2}{\partial y^2}\right).$$

Again, discarding the term involving $\mathcal{A}^-$, we see that (20) reduces to

$$\frac{\partial \psi}{\partial x} = \frac{\mathrm{i}}{2k}\left(2k^2 + \mathrm{i}\frac{\partial k}{\partial x} + \frac{\partial^2}{\partial y^2}\right)\psi. \tag{22}$$

This equation was obtained by Corones [35]. It is the equation studied by Radder [9] for the propagation of water waves. If we rewrite (22) using (11), we obtain

$$2\mathrm{i}k\frac{\partial u}{\partial x} + 2k(k-k_0)u + \mathrm{i}u\frac{\partial k}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{23}$$

as the equation governing the amplitude $u(\boldsymbol{x})$.

A third choice for T is

$$T_3 = \tfrac{1}{2}\begin{pmatrix} 1 & -\mathrm{i}\mathcal{P}^{-1} \\ 1 & \mathrm{i}\mathcal{P}^{-1} \end{pmatrix},$$

whence

$$\mathcal{A}^- = \tfrac{1}{2}\mathcal{P}^{-1}\frac{\partial}{\partial x}\mathcal{P}.$$

This choice has the virtue that the equations for ψ^+ and ψ^- decouple when k is independent of x. Discarding $\mathcal{A}^-$, we obtain

$$\frac{\partial \psi}{\partial x} = \mathrm{i}\mathcal{P}\psi, \tag{24}$$

an equation due to Claerbout [36]. Indeed, this equation follows from a simple factorization of the Helmholtz equation,

$$\frac{\partial^2}{\partial x^2} + \mathcal{P}^2 = \left(\frac{\partial}{\partial x} + \mathrm{i}\mathcal{P}\right)\left(\frac{\partial}{\partial x} - \mathrm{i}\mathcal{P}\right),$$

assuming that k does not vary with x [7].

In order to use an equation such as (24), we must be able to compute the *square-root operator* $\mathcal{P}$, defined formally by

$$\mathcal{P} = \sqrt{k^2 + \frac{\partial^2}{\partial y^2}}.$$

It is known that $\mathcal{P}$ is a non-local operator, which means that it cannot be written as a differential operator, even if k is constant. In fact, $\mathcal{P}$ is a pseudodifferential operator. However, $\mathcal{P}$ can be approximated using differential operators. This can be seen most clearly using Fourier analysis when k is constant. We sketch this approach below; McCoy [7] discusses the case where k is independent of x. However, formal approximations of $\mathcal{P}$ are often made. For example, Tappert [2, p. 276] derives the equation

$$\frac{\partial \psi}{\partial x} = \frac{\mathrm{i}}{2}\frac{\partial^*}{\partial y}\left(\frac{1}{k}\frac{\partial \psi}{\partial y}\right) + \mathrm{i}k\psi - \frac{\mathrm{i}}{4k}\frac{\partial}{\partial y}\left(\frac{1}{k}\frac{\partial k}{\partial y}\right)\psi,$$

which Ames & Lee [4] call the *range refraction parabolic wave equation*; see also [37].

3.3 Computation of the reflected wave

The splitting method described in the previous section does not automatically eliminate consideration of a reflected wave component ψ^-; in fact, we saw that the decoupling of the equations is usually imperfect and is achieved only as an assumption about the relatively weak effect of a reflected wave on the forward propagating component. There have been a few instances in the literature where a computation of the reflected wave is also made based on the complete system (20) and (21). For the mildly-sloped bottoms considered here, reflection is usually weak unless some undular feature of the bed causes a resonance between incident and reflected wave components.

Corones [35] extended the methods of Bremmer [38] and Bellman & Kalaba [39] to the coupled system (20) and (21). First, rewrite the system using parabolic differential operators $P^{\pm}(\boldsymbol{x})$ given by

$$\begin{aligned} P^+(\boldsymbol{x}) &= \frac{\partial}{\partial x} - \mathcal{A}^+(\boldsymbol{x}), \\ P^-(\boldsymbol{x}) &= \frac{\partial}{\partial x} - \mathcal{B}^-(\boldsymbol{x}), \end{aligned}$$

to obtain

$$P^+\psi^+ = \mathcal{A}^-\psi^- \quad \text{and} \quad P^-\psi^- = \mathcal{B}^+\psi^+. \tag{25}$$

If we then let $G^{\pm}(\boldsymbol{x}|\boldsymbol{x}')$ denote the Green's function of the parabolic differential operators $P^{\pm}$, we may write the solution to (25) as

$$\begin{aligned}\psi^+(\boldsymbol{x}) &= \psi_0^+(\boldsymbol{x}) + \int G^+(\boldsymbol{x}|\boldsymbol{x}')\,\mathcal{A}^-(\boldsymbol{x}')\,\psi^-(\boldsymbol{x}')\,d\boldsymbol{x}',\\ \psi^-(\boldsymbol{x}) &= \psi_0^-(\boldsymbol{x}) + \int G^-(\boldsymbol{x}|\boldsymbol{x}')\,\mathcal{B}^+(\boldsymbol{x}')\,\psi^+(\boldsymbol{x}')\,d\boldsymbol{x}',\end{aligned}$$

where $\psi_0^{\pm}$ are the decoupled, forward and backward-scattered solutions to the parabolic equations

$$P^{\pm}\psi_0^{\pm} = 0, \tag{26}$$

and where suitable boundary conditions have to to prescribed to initialize the forward and backward propagating waves at the beginning and end of the domain. Iteration of these integral equations yields the Bremmer series.

One question of concern in the Bremmer series representation is the convergence of the resulting series itself. Atkinson [40] showed that the convergence of the series is strictly limited by the smallness of the absolute integral of the reflection operators ($\mathcal{A}^-$, $\mathcal{B}^+$) over the range of the depth inhomogeneity. There is not enough evidence in the literature to deduce whether this convergence limit is a practical hindrance in situations where bottoms take on naturally mild configurations. Most applications of the ideas here have used an iterative approach in order to obtain computational results; see, for example, Liu & Tsay [41]. To illustrate such a method, consider the system (25). First, compute an initial condition for the iteration using (26). Then, a sequence of iterates $\psi_{(n)}^{\pm}$ is constructed using the formulae

$$P^+\psi_{(n)}^+ = \mathcal{A}^-\psi_{(n-1)}^- \quad \text{and} \quad P^-\psi_{(n)}^- = \mathcal{B}^+\psi_{(n-1)}^+.$$

For situations where only weak reflections arise, the iteration proceeds to rapid convergence in as few as two steps [41]. However, in a study of resonant reflection from undular bed forms, Kirby [28] found that the iteration process suggested here also diverges if the initial estimate of the reflected wave is too large, and found it necessary to modify the present technique using an under-relaxation scheme.

3.4 Fourier analysis

When $k = k_0$, we can use Fourier analysis to represent solutions of (10) in terms of plane waves:

$$\psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\psi}(\ell,m)\,\mathrm{e}^{\mathrm{i}(\ell x+my)}\,d\ell\,dm.$$

Substituting into (10) shows that the only allowable values of ℓ and m are those satisfying

$$a(\ell,m) \equiv k_0^2 - \ell^2 - m^2 = 0, \tag{27}$$

that is

$$\ell = k_0\cos\theta \qquad \text{and} \qquad m = k_0\sin\theta, \qquad -\pi \le \theta < \pi,$$

corresponding to plane waves propagating in all directions θ; equation (27) is the equation of a circle in the (ℓ,m)-plane.

The function $a(\ell,m)$ is called the *symbol* of the differential operator $\nabla^2 + k_0^2$. Elementary properties of the Fourier transform show that differential operators have polynomial symbols.

Now, we want an equation governing propagation in the forward direction, that is, $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. Thus, we want the semicircle in $\ell > 0$, namely,

$$\ell = +\sqrt{k_0^2 - m^2}. \tag{28}$$

This does not correspond to a differential operator. However, we can approximate the square-root in various ways, so as to obtain differential equations.

For good accuracy close to the x-axis, we can assume that m is small; this gives the approximation

$$\ell = k_0\{1 - \tfrac{1}{2}(m/k_0)^2\},$$

a parabola in the (ℓ,m)-plane. It corresponds to the simple parabolic approximation (16).

Another possibility is to approximate the square-root by a rational function; following Kirby [42], write

$$\sqrt{k_0^2 - m^2} \simeq k_0\frac{a_0 + a_1(m/k_0)^2}{1 + b_1(m/k_0)^2}. \tag{29}$$

If we expand both sides in powers of m^2 and match terms up to m^4, we find that

$$a_0 = 1, \qquad a_1 = -\tfrac{3}{4} \qquad \text{and} \qquad b_1 = -\tfrac{1}{4}. \tag{30}$$

Alternatively, we can determine the coefficients a_0, a_1 and b_1 by matching both sides of (29) in an average (minimax) sense over a specified range of m. Such approximations are discussed in [42] and [43]. These papers also consider higher-order approximations; for some comparisons with shallow-water theory applied to a simple step, see [44].

Let us take $a_0 = 1$, so that a plane wave can propagate along the x-axis without distortion. Then, equations (28) and (29) give the relation

$$\mathrm{i}\ell\{1 + b_1(m/k_0)^2\} = \mathrm{i}k_0\{1 + a_1(m/k_0)^2\}$$

in Fourier space. In physical space, this corresponds to

$$\frac{\partial \psi}{\partial x} - \frac{b_1}{k_0^2}\frac{\partial^3 \psi}{\partial x\,\partial y^2} = \mathrm{i}k_0\psi - \frac{\mathrm{i}a_1}{k_0}\frac{\partial^2 \psi}{\partial y^2}.$$

Making the substitution (11) gives

$$2\mathrm{i}k_0\frac{\partial u}{\partial x} + 2\alpha_0\frac{\partial^2 u}{\partial y^2} + \frac{2\mathrm{i}\alpha_1}{k_0}\frac{\partial^3 u}{\partial x\,\partial y^2} = 0, \tag{31}$$

where $\alpha_0 = b_1 - a_1$ and $\alpha_1 = -b_1$ are constants at our disposal; we assume that they are non-negative. We call (31) the *wide-angle parabolic equation*. It reduces to the simple parabolic equation (15) when $\alpha_0 = \frac{1}{2}$ and $\alpha_1 = 0$. When the choices (30) are made, it becomes

$$2\mathrm{i}k_0\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\mathrm{i}}{2k_0}\frac{\partial^3 u}{\partial x\,\partial y^2} = 0,$$

which is known as *Claerbout's equation* [3, pp. 206–207].

4 Numerical representation of the simple parabolic equation

In the remainder of this paper, we shall concentrate mainly on the simple parabolic equation, (15). We start by describing the basic discretization of this equation using the Crank–Nicolson scheme. We consider a (semi-infinite) rectangular computational domain

$$\mathcal{C}_b = \{(x, y) : x > 0,\ 0 < y < b\}.$$

An 'initial condition' is prescribed on the end $x = 0$, $0 < y < b$. Lateral boundary conditions are prescribed on the two sides, $y = 0$ and $y = b$, for $x > 0$. These conditions will be discussed in detail later. As the governing partial differential equation is parabolic, it is not necessary to specify a 'downwave condition' at large values of x.

We superimpose a regular mesh on the computational domain $\mathcal{C}_b$, with grid points at (x_j, y_n), where $x_j = j\Delta x$, $y_n = (n - \frac{1}{2})\Delta y$, $j = 0, 1, 2, \ldots, J$ and $n = 0, 1, 2, \ldots, N$. By construction, $b = (N - 1)\Delta y$. Denote

$$u_n^j = u(x_j, y_n).$$

The Crank–Nicolson approach, which is an implicit scheme with second-order accuracy in both Δx and Δy, is written as

$$2\mathrm{i}k_0\frac{u_n^j - u_n^{j-1}}{\Delta x} + \tfrac{1}{2}\left\{\frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{(\Delta y)^2} + \frac{u_{n+1}^{j-1} - 2u_n^{j-1} + u_{n-1}^{j-1}}{(\Delta y)^2}\right\} = 0.$$

This scheme is consistent and stable. It is also convenient in that it uses only the results from row $j - 1$ to compute row j, resulting in a tridiagonal matrix when applied to all of the n values excluding those on the boundary. Boundary conditions have to be applied at $n = 0$ and $n = N$ to provide enough equations to solve for the unknowns, u_n^j, $n = 0, 1, 2, \ldots, N$. Once these conditions are included, the resulting equations may be solved using a (complex) tridiagonal solver; these are very fast. An analysis of the numerical scheme (in fact, for the wide-angle parabolic equation (31) in the context of underwater acoustics) has been given by St. Mary & Lee [45].

5 Lateral boundary conditions: overview

Typical numerical implementations of parabolic models use very simple lateral boundary conditions, such as impedance conditions. These lateral boundary conditions are imperfect, in that they reflect waves back into the computational domain. Hence, a very wide domain must be used so that the influence of the lateral boundary conditions is far away from the region of interest, so as to not introduce any numerical contamination. This of course means that far more numerical computation is carried out than is desired. Efficient lateral boundary conditions would mean that model computations would only include the region of interest.

In the context of water waves, two problems are of interest. These are the diffraction and transmission problems. More precisely, divide the first quadrant ($x > 0$, $y > 0$) into an 'illuminated region' ($x > 0$, $0 < y < b$) and a 'shadow region' ($x > 0$, $y > b$).

For a *diffraction problem*, suppose that there is a semi-infinite breakwater along $x = 0$, $y > b$ (where u vanishes) and waves are incident from the region $x < 0$. In standard implementations, computations would include both the the illuminated region and (part of) the shadow region. If only the illuminated region is of interest, we could reduce the computational domain to that region by placing a suitable *diffractive boundary condition* along the interface between the two regions, namely $x > 0$, $y = b$; see §7.1.

For a *transmission problem*, we allow waves to pass cleanly through the line $x = 0$, $y > b$ (along which u does not vanish, in general). In order to reduce the computational domain to the illuminated region, we now place a suitable *transmitting boundary condition* along $x > 0$, $y = b$; see §7.2.

Let us now describe some of the lateral boundary conditions usually employed in the literature. First, we note that one solution of (15) is

$$u(x,y) = \mathrm{e}^{-\frac{1}{2}\mathrm{i}k_0 x \sin^2\theta}\mathrm{e}^{\mathrm{i}k_0(y-b)\sin\theta},$$

corresponding to a plane wave propagating at an angle θ to the x-axis. For the case of reflection from a lateral boundary at $y = b$, we can write the total solution as

$$u(x,y) = \mathrm{e}^{-\frac{1}{2}\mathrm{i}k_0 x \sin^2\theta}\left(\mathrm{e}^{\mathrm{i}k_0(y-b)\sin\theta} + R\,\mathrm{e}^{-\mathrm{i}k_0(y-b)\sin\theta}\right), \tag{32}$$

where R is the (complex) reflection coefficient; $|R|$ can vary between zero and unity depending on the amount of reflection from the boundary.

Now, the most common lateral boundary condition used with parabolic models is an impedance boundary condition. The impedance, $\mathcal{Z}$, in our context, is defined as the ratio of pressure at the boundary to normal velocity at the boundary,

$$\mathcal{Z} = \frac{\rho\partial\Phi/\partial t}{-\partial\Phi/\partial y} = \frac{\mathrm{i}\omega\rho\phi}{\partial\phi/\partial y} \qquad \text{on } y = b, \tag{33}$$

where ρ is the density. For our plane-wave solution with reflection, (32), we have

$$\mathcal{Z} = \frac{\rho\omega}{k_0\sin\theta}\frac{(1+R)}{(1-R)} \tag{34}$$

showing that the impedance of the boundary depends on the angle of incidence and the reflection coefficient; usually, neither of these is known.

Rewriting (33), using (2) and

$$\zeta(\boldsymbol{x}) = u(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}k_0 x},$$

we have the impedance boundary condition

$$\frac{\partial u}{\partial y} - \frac{\mathrm{i}\rho\omega}{\mathcal{Z}}u = 0 \qquad \text{on } y = b. \tag{35}$$

In general, $\mathcal{Z}$ could be real or complex. A real value of the impedance will lead to transmission of waves and wave energy. A purely imaginary value will result in a phase shift of the reflected wave, but no transmission. A complex $\mathcal{Z}$ results in both transmission and reflection.

For example, for plane waves with the impedance given as (34), the impedance boundary condition becomes

$$\frac{\partial u}{\partial y} - \mathrm{i}k_0\sin\theta\,\frac{(1-R)}{(1+R)}u = 0 \qquad \text{on } y = b.$$

For a perfectly reflecting boundary, R is set to unity. For a perfectly transmitting boundary condition, R is set to zero, yielding

$$\frac{\partial u}{\partial y} - \mathrm{i}k_0 u\sin\theta = 0 \qquad \text{on } y = b. \tag{36}$$

In addition to planar wave trains at the boundary, this boundary condition requires that both k_0 and the wave angle, θ, be known at the boundary. These parameters can be difficult to obtain within a computational model (especially if $h(\boldsymbol{x})$ varies in $\mathcal{C}_b$).

Kirby [46] has used a numerical implementation of (36), for waves propagating over a variable-depth domain. He assumes that the longshore wavenumber, $k_0\sin\theta$, is calculable from the previous computational grid row. In finite-difference form, the boundary condition becomes

$$\frac{u^j_{n+1} - u^j_n}{\Delta y} - \mathrm{i}k_0\sin\theta\left(\frac{u^j_{n+1} + u^j_n}{2}\right) = 0, \tag{37}$$

for $n = 0$ or $n = N-1$, when the boundary is located at $y = 0$ or $y = b$, respectively. $k_0\sin\theta$ is found by evaluating (37) at the previous grid row:

$$k_0\sin\theta = -\frac{2\mathrm{i}}{\Delta y}\frac{\left(u^{j-1}_{n+1} - u^{j-1}_n\right)}{\left(u^{j-1}_{n+1} + u^{j-1}_n\right)}. \tag{38}$$

Kirby [46] has shown that this condition, (37) with (38), is exact for plane waves. It may be used on the upwave or downwave lateral boundary, but should be far from scattering objects within the model domain, as scattering of waves occurring within the computational domain will be partially reflected by this 'plane-wave' boundary condition. The intention of this practice is to have the weak reflection enter the computational domain downwave of the area of interest.

For an arbitrary impedance, Dalrymple [47] examined the decay of waves in a entrance channel with rubblemound jetties. The wave height, due to the diffraction of wave energy into the dissipating sidewalls of the channel, was shown to decay exponentially down the channel.

Other lateral boundary conditions have been used. For example, 'sponge-layer' models have been introduced, which involve regions of high energy dissipation near the side walls of the computational domain [48], [49].

Dalrymple & Martin [50] have developed some generalized impedance boundary conditions which permit waves to leave the computational domain regardless of the wave direction, crest curvature, or strength of scattering. They are *perfect boundary conditions*, because they are exact, apart from discretization errors. The idea is to solve the governing differential equation in the 'shadow region' exactly, using integral transforms. This leads to an exact condition which is to be imposed on the lateral boundary. This idea was used by Marcus [51] to develop transmitting boundary conditions for underwater acoustics; he used Fourier transforms. We use Laplace transforms, as these are more convenient. We obtain appropriate conditions for diffracting as well as transmitting boundaries. Numerical comparisons with Kirby's plane-wave boundary condition have been made [50]; these show the efficacy of the new boundary condition. The method can be extended so as to treat the wide-angle equation (31).

Givoli [52] has reviewed the extensive literature on 'non-reflecting boundary conditions', although he does not consider the specific problem of deriving good lateral boundary conditions for parabolic equations. Our exact boundary conditions are non-local and similar to [52, equation (26)]; they become computationally useful after appropriate discretization.

6 Exact solution in the shadow region

In order to derive perfect lateral boundary conditions, we solve the simple parabolic equation (15) exactly within the 'shadow region', $x > 0$, $y > b$, subject to appropriate boundary conditions. Thus, we consider

$$\frac{\partial^2 U}{\partial y^2} + \pi\Omega^2 \frac{\partial U}{\partial x} = 0 \tag{39}$$

subject to the boundary conditions

$$U(x, b) = u_b(x) \qquad \text{for } x > 0 \tag{40}$$

and

$$U(0, y) = 0 \qquad \text{for } y > b, \tag{41}$$

where $u_b(x)$ is assumed known and

$$\Omega^2 = 2ik_0/\pi; \tag{42}$$

we also assume that

$$U(x, y) \text{ is bounded as } y \to \infty. \tag{43}$$

Clearly, the solution of this problem is a function of $(y - b)$, so, without loss of generality, we can set $b = 0$.

We note that, due to the assumption (12), (41) is a comparable approximation to $\partial\phi/\partial x = 0$ on $x = 0$, which is itself the appropriate boundary condition on a rigid wall or impermeable breakwater.

We solve for U using a Laplace transform in x:

$$\mathcal{L}\{U\} \equiv \overline{U}(p, y) = \int_0^\infty U(x, y)\, \mathrm{e}^{-px}\, dx, \tag{44}$$

where we suppose that $\mathrm{Re}\, p > 0$. Since

$$\mathcal{L}\left\{\frac{\partial U}{\partial x}\right\} = p\overline{U}(p, y) - U(0, y) = p\overline{U}(p, y),$$

by (41), (39) is transformed into

$$\frac{\partial^2 \overline{U}}{\partial y^2} + \pi p \Omega^2 \overline{U} = 0,$$

with general solution

$$\overline{U}(p,y) = C(p)\exp\{\mathrm{i}y\Omega\sqrt{\pi p}\} + D(p)\exp\{-\mathrm{i}y\Omega\sqrt{\pi p}\}.$$

Given (42), we define Ω by

$$\Omega = (1+\mathrm{i})\sqrt{(k_0/\pi)}.$$

Then, (43) implies that $D(p) \equiv 0$, whence (40) gives $C(p) = \overline{u_b}(p)$. Hence,

$$\overline{U}(p,y) = \overline{u_b}(p)\exp\{(-1+\mathrm{i})y\sqrt{k_0 p}\}. \tag{45}$$

This formula can be inverted using the convolution theorem, namely

$$\mathcal{L}\{u\}\mathcal{L}\{v\} = \mathcal{L}\left\{\int_0^x u(x-\xi)v(\xi)\,d\xi\right\}.$$

From [53, §17.13, equation (32)], we have

$$\exp\{\mathrm{i}y\Omega\sqrt{\pi p}\} = \mathcal{L}\left\{-\frac{\mathrm{i}y\Omega}{2x^{3/2}}\exp\left(\frac{\pi\Omega^2y^2}{4x}\right)\right\}.$$

Hence, the convolution theorem gives

$$U(x,y) = -\tfrac{1}{2}\mathrm{i}y\Omega\int_0^x \frac{u_b(\xi)}{(x-\xi)^{3/2}}\exp\left\{\frac{\mathrm{i}k_0y^2}{2(x-\xi)}\right\}d\xi. \tag{46}$$

This function is small for large y, as required: an integration by parts shows that, as $y\to\infty$,

$$U(x,y) \sim \frac{2\mathrm{i}x^{1/2}}{\pi\Omega y}u_b(0)\exp\left\{\frac{\mathrm{i}k_0y^2}{2x}\right\} + O(y^{-3}).$$

If we differentiate (45) with respect to y, we obtain

$$\frac{\partial\overline{U}(p,y)}{\partial y} = \mathrm{i}\Omega\sqrt{\pi p}\,\overline{U}(p,y) = \mathrm{i}\Omega\sqrt{(\pi/p)}\,\{p\overline{U}(p,y)\}. \tag{47}$$

Using the convolution theorem again, we deduce that

$$\frac{\partial U(x,y)}{\partial y} = \mathrm{i}\Omega\int_0^x \frac{\partial U(\xi,y)}{\partial\xi}\frac{d\xi}{\sqrt{x-\xi}}; \tag{48}$$

in particular, on $y=0$,

$$\frac{\partial U(x,0)}{\partial y} = \mathrm{i}\Omega\int_0^x \frac{u_b'(\xi)}{\sqrt{x-\xi}}\,d\xi. \tag{49}$$

This equation is the starting point for the derivation of the generalized impedance boundary condition. Note that it is easier to derive (49) by working in the transform domain, rather than by manipulating the formula (46).

7 Perfect boundary conditions

We have just given an exact description of the waves in the shadow region (outside of the computational domain); denote the solution in this region by $U(x,y)$. Denote the solution in the computational domain (the illuminated region) by $u(x,y)$. These two solutions must match across the interface $y=b$,

$$u(x,b) = U(x,b) \qquad \text{and} \qquad \frac{\partial u(x,b)}{\partial y} = \frac{\partial U(x,b)}{\partial y}. \tag{50}$$

Now, for diffraction problems (see §5), U is given by (46) as

$$U(x,y) = -\tfrac{1}{2}\mathrm{i}(y-b)\Omega\int_0^x \frac{U(\xi,b)}{(x-\xi)^{3/2}}\exp\left\{\frac{\mathrm{i}k_0(y-b)^2}{2(x-\xi)}\right\}d\xi$$

for $y \geq b$. Although this formula shows that $U(x,y)$ depends solely on values of $U(\xi,b)$ for $\xi \leq x$, it is not easy to use within standard numerical schemes. Instead, we use (48); if we set $y=b$ therein, and use the matching conditions (50), we obtain

$$\frac{\partial u(x,b)}{\partial y} = \mathrm{i}\Omega\int_0^x \frac{\partial u(\xi,b)}{\partial\xi}\frac{d\xi}{\sqrt{x-\xi}}. \tag{51}$$

The formula (51) is exact. It shows that the condition to be imposed on $y=b$ is *non-local* and *not* an impedance condition of the form (35). However, when (51) is discretized, it leads to a condition that is similar to an inhomogeneous impedance condition; it can be called a *generalized impedance boundary condition.*

From (51), we have

$$\frac{\partial u_b^j}{\partial y} = \mathrm{i}\Omega\int_0^{x_j} \frac{\partial u(\xi,b)}{\partial\xi}\frac{d\xi}{\sqrt{x_j-\xi}},$$

where $u_b^j = u(x_j,b)$. We assume that $u(\xi,b)$ is approximated by a continuous piecewise-linear function, so that

$$\frac{\partial u(\xi,b)}{\partial\xi} = \frac{u_b^{l+1}-u_b^l}{\Delta x} \qquad \text{for } x_l < \xi < x_{l+1}.$$

Hence

$$\frac{\partial u_b^j}{\partial y} = \sum_{l=0}^{j-1}(u_b^{l+1}-u_b^l)\,L_l(x_j), \tag{52}$$

where

$$L_l(x) = \frac{\mathrm{i}\Omega}{\Delta x}\int_{x_l}^{x_{l+1}} \frac{d\xi}{\sqrt{x-\xi}} = \frac{2\mathrm{i}\Omega}{\Delta x}\left(\sqrt{x-x_l} - \sqrt{x-x_{l+1}}\right),$$

whence

$$L_l(x_j) = \frac{2\mathrm{i}\Omega}{\sqrt{\Delta x}}\left(\sqrt{j-l} - \sqrt{j-l-1}\right).$$

Rearranging (52), we have

$$\begin{aligned}\frac{\partial u_b^j}{\partial y} - u_b^j L_{j-1}(x_j) &= -u_b^{j-1}L_{j-1}(x_j) + \sum_{l=0}^{j-2}(u_b^{l+1} - u_b^l)\,L_l(x_j) \\ &= \sum_{l=1}^{j-1} u_b^l\,\{L_{l-1}(x_j) - L_l(x_j)\} - u_b^0 L_0(x_j).\end{aligned}$$

If we substitute for $L_l(x_j)$, we obtain

$$\frac{\partial u_b^j}{\partial y} + a u_b^j = \sum_{l=0}^{j-1} b_l^j u_b^l \tag{53}$$

as our generalized impedance boundary condition on $y = b$, where

$$a = -\frac{2\mathrm{i}\Omega}{\sqrt{\Delta x}}, \tag{54}$$

$$b_0^j = -\frac{2\mathrm{i}\Omega}{\sqrt{\Delta x}}\left(\sqrt{j} - \sqrt{j-1}\right), \tag{55}$$

$$b_l^j = -\frac{2\mathrm{i}\Omega}{\sqrt{\Delta x}}\left(2\sqrt{j-l} - \sqrt{j-l-1} - \sqrt{j-l+1}\right), \tag{56}$$

for $l = 1, 2, \ldots, j-1$.

7.1 The diffractive boundary condition

The analysis above is appropriate when the lateral boundary at $y = b$ divides an illuminated region ($y < b$), where the wave field will be calculated numerically, and a shadow region ($y > b$) into which the waves are diffracting. We call the corresponding condition on $y = b$ the *diffractive boundary condition.* It is implemented into the parabolic model by

recalling that this boundary is at $y = \frac{1}{2}(y_{N-1} + y_N)$; thus, we use central differences and averages to approximate $\partial u_b^j/\partial y$ and u_b^j, respectively. The result is

$$\frac{u_N^j - u_{N-1}^j}{\Delta y} + \frac{a}{2}\left(u_N^j + u_{N-1}^j\right) = \tfrac{1}{2}\sum_{l=0}^{j-1} b_l^j(u_N^l + u_{N-1}^l).$$

This is the diffractive boundary condition; it is used for $n = N$ in the matrix formulation of the problem.

7.2 The transmitting boundary condition

Suppose that we have a transmission problem (see §5) with a known incident wave field $u_{\mathrm{inc}}(x, y)$; in general, $u_{\mathrm{inc}}(0, y)$ does not vanish identically for $y > b$. Let us perturb the incident field within the computational domain ($y < b$), so that the total field is $u(x, y)$. The difference,

$$u - u_{\mathrm{inc}} = u_{\mathrm{sc}}, \tag{57}$$

say, will satisfy $u_{\mathrm{sc}}(0, y) = 0$ for $y > b$, and hence will solve the same mathematical problem in the shadow region as U. Thus,

$$\frac{\partial u_{\mathrm{sc}}(x,b)}{\partial y} = \mathrm{i}\Omega\int_0^x \frac{\partial u_{\mathrm{sc}}(\xi,b)}{\partial \xi}\frac{d\xi}{\sqrt{x-\xi}}. \tag{58}$$

It follows that we can derive a boundary condition for u_{sc} on $y = b$ by discretization, as for the diffractive boundary condition.

In practice, a boundary condition for u is often preferable. Combining equations (57) and (58), we obtain

$$\frac{\partial u(x,b)}{\partial y} = \mathrm{i}\Omega\int_0^x \frac{\partial u(\xi,b)}{\partial \xi}\frac{d\xi}{\sqrt{x-\xi}} + F_{\mathrm{inc}}(x),$$

where

$$F_{\mathrm{inc}}(x) = \frac{\partial u_{\mathrm{inc}}(x,b)}{\partial y} - \mathrm{i}\Omega\int_0^x \frac{\partial u_{\mathrm{inc}}(\xi,b)}{\partial \xi}\frac{d\xi}{\sqrt{x-\xi}} \tag{59}$$

is known, in principle. If u_{inc} is only known numerically, (59) can be discretized as before. If u_{inc} is known analytically, further progress may be possible [50].

In summary, we have two alternative transmitting boundary conditions, one for u_{sc} (the change in u_{inc} due to any scattering from the computational domain) and one for u (the total field, namely the sum of u_{inc} and u_{sc}).

7.3 Results and extensions

To illustrate the use of the perfect boundary conditions, a numerical model was set up with Kirby's transmitting condition (37) at $y = 0$ and a perfect boundary condition at $y = b$. Various incident fields were used, and both diffraction and transmission problems were solved. In all cases, the waves were observed to transmit through $y = b$ with only negligible reflection. The boundary at $y = 0$ induced spurious reflections (except for incident plane waves). These numerical results, described in [50], suggest that the perfect boundary conditions are efficient and effective. In practice, the conditions would be used between a constant-depth region (shadow region) and a variable-depth region (computational domain). Application to variable water depths in both domains is straightforward, through the use of a variable transformation along the boundary, as used by Liu & Mei [8].

Perfect boundary conditions can also be developed for the wide-angle parabolic equation, (31). The key result is that the wave field in the shadow region is exactly modelled by [50]

$$\frac{\partial u(x,b)}{\partial y} = \frac{\mathrm{i}k_0}{\sqrt{\alpha_1}} \int_0^x \mathrm{e}^{\mathrm{i}\lambda(x-\xi)} J_0(\lambda(x-\xi)) \frac{\partial u(\xi,b)}{\partial \xi}\, d\xi,$$

where J_0 is a Bessel function and

$$\lambda = \frac{k_0\alpha_0}{2\alpha_1}.$$

This can then be discretized as before; see [50] for further details.

8 Porous regions

In this section, we generalize the analysis above so as to treat the propagation of water waves in regions bounded laterally by porous media, which we model using the equations of Sollitt & Cross [54]. The simplest problem of interest is when waves are obliquely incident on a thick porous breakwater lying along the x-axis. We derive a perfect boundary condition for use in numerical models.

To be specific, we can consider the following diffraction problem, in which there is a thin rigid semi-infinite breakwater along $x = 0$, $y > 0$. We suppose that the second, third and fourth quadrants in the (x,y)-plane are filled with water, and that the first quadrant is filled with a

porous medium. We suppose further that waves are incident from the region $x < 0$; thus, the exact incident field is given by

$$\zeta_{\mathrm{inc}}(x,y) = \mathrm{e}^{\mathrm{i}k_0(x\cos\theta + y\sin\theta)}.$$

The incident waves are reflected by the rigid face of the breakwater ($x = 0$, $y > 0$), diffracted by the corner at the origin, and refracted into the porous quadrant. The exact solution, then, is governed by (10) in the water, namely

$$(\nabla^2 + k_0^2)\zeta = 0,$$

a different Helmholtz equation in the porous medium, namely

$$(\nabla^2 + K_1^2)Z = 0$$

(K_1 is defined below), continuity conditions relating ζ and Z across the interface ($y = 0$, $x > 0$), and zero-velocity conditions on the breakwater ($x = 0$, $y > 0$). This problem has been formulated by Meister [55], but it has not been solved.

To make progress, we start by invoking a Kirchhoff approximation [56, Chap. 8]. Thus, we suppose that $\zeta(0,y)$ is known for $y < 0$, and then try to calculate the transmitted field in $x > 0$. Usually, one assumes that $\zeta(0,y) = \zeta_{\mathrm{inc}}(0,y)$ for $y < 0$, but one could suppose that $\zeta(0,y)$ is known from experimental measurements. This leads to a problem posed in the half-plane $x > 0$, which is itself composed of two quadrants, one filled with water and one filled with the porous medium. This problem is still complicated, although it has been studied previously [57]. However, the generalization to more complicated problems, such as a porous-walled channel, seems to be intractable.

To make further progress, we retain the Kirchhoff approximation, but also invoke the parabolic approximation. Thus, the incident field is now given by

$$u_{\mathrm{inc}}(x,y) = \mathrm{e}^{-\gamma x + \mathrm{i}my}, \tag{60}$$

where $\zeta_{\mathrm{inc}}(x,y) = u_{\mathrm{inc}}(x,y)\,\mathrm{e}^{\mathrm{i}k_0 x}$,

$$\gamma = \tfrac{1}{2}\mathrm{i}k_0\sin^2\theta \qquad \text{and} \qquad m = k_0\sin\theta.$$

We aim to calculate the wave field in $x > 0$, using simple parabolic models.

Writing $\zeta = u\,\mathrm{e}^{\mathrm{i}k_0 x}$, as before, we obtain

$$\frac{\partial^2 u}{\partial y^2} + \pi\Omega^2\frac{\partial u}{\partial x} = 0 \qquad \text{in } Q_-,$$

where $Q_- = \{(x,y) : x > 0,\ y < 0\}$ and $\Omega^2 = 2\mathrm{i}k_0/\pi$.

We suppose that the first quadrant $Q_+ = \{(x,y) : x > 0,\ y > 0\}$ is occupied by a rigid porous medium. The fluid motion within Q_+ may also be described using a potential and a modified free-surface boundary condition. These equations have been derived by Sollitt & Cross [54]; see also [58, Appendix A]. The porous medium is characterised by three parameters: the porosity, ϵ, the linear friction factor, f, and the inertial term, s; all these parameters are taken to be constant here. All wave motion in Q_+ is damped if $f > 0$. For our parabolic model in Q_+, we choose the 'least-damped' wavenumber, K_1; this is the root of the complex dispersion relation,

$$\omega^2(s + \mathrm{i}f) = gK_1 \tanh K_1 h,$$

in the first quadrant of the complex K_1-plane with smallest imaginary part. To get a parabolic approximation, we write

$$Z(\boldsymbol{x}) = U(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}K_1 x},$$

whence U satisfies

$$\frac{\partial^2 U}{\partial y^2} + \pi\Omega_1^2 \frac{\partial U}{\partial x} = 0 \qquad \text{in } Q_+, \tag{61}$$

where

$$\Omega_1^2 = 2\mathrm{i}K_1/\pi. \tag{62}$$

The boundary conditions on $x = 0$ are

$$u(0,y) = u_{\mathrm{inc}}(0,y) = \mathrm{e}^{\mathrm{i}my} \qquad \text{for } y < 0, \text{ and}$$

$$U(0,y) = 0 \qquad \text{for } y > 0, \tag{63}$$

where we have used (60). There are also continuity conditions across the interface between the water and the porous medium. These are [58]

$$\partial\zeta/\partial y = \epsilon\partial Z/\partial y \qquad \text{and} \qquad \zeta = (s + \mathrm{i}f)Z \qquad \text{on } y = 0,\ x > 0.$$

In terms of u and U, these become

$$\mathrm{e}^{\mathrm{i}\kappa x}\,\partial u/\partial y = \epsilon\,\partial U/\partial y \qquad \text{and} \qquad u\,\mathrm{e}^{\mathrm{i}\kappa x} = (s + \mathrm{i}f)U \tag{64}$$

on $y = 0$, $x > 0$, where

$$\kappa = k_0 - K_1.$$

Finally, we also assume that u and U are bounded for large $|y|$.

8.1 Solution in the porous region Q_+

We apply the Laplace transform in x, defined by (44), to the differential equation (61). Making use of (63), we obtain

$$\frac{\partial^2 \overline{U}}{\partial y^2} + \pi p \Omega_1^2 \overline{U} = 0,$$

which has the general solution

$$\overline{U}(p,y) = C(p)\exp\{\mathrm{i}y\Omega_1\sqrt{\pi p}\} + D(p)\exp\{-\mathrm{i}y\Omega_1\sqrt{\pi p}\}.$$

Now, from the definition of the complex wavenumber K_1, we have

$$K_1 = |K_1|\,\mathrm{e}^{\mathrm{i}\delta} \qquad \text{with} \qquad 0 \le \delta < \pi/2,$$

and so, given (62), we define Ω_1 by

$$\Omega_1 = (1 + \mathrm{i})\sqrt{|K_1|/\pi}\,\mathrm{e}^{\mathrm{i}\delta/2}.$$

Then, we must have $D(p) \equiv 0$ for boundedness as $y \to \infty$. Hence, on $y = 0$, we have

$$\begin{aligned} \overline{U}(p,0) &= C(p) \qquad \text{and} \\ \partial\overline{U}/\partial y &= \mathrm{i}\Omega_1\sqrt{\pi p}\,C(p). \end{aligned}$$

In fact, for all $y > 0$, we have

$$\partial\overline{U}/\partial y = \mathrm{i}\Omega_1\sqrt{\pi p}\,\overline{U}.$$

This is exactly the formula (47), with Ω replaced by Ω_1. Inverting and setting $y = 0$, we obtain equation (49):

$$\frac{\partial U(x,0)}{\partial y} = \mathrm{i}\Omega_1 \int_0^x \frac{\partial U(\xi,0)}{\partial \xi}\,\frac{d\xi}{\sqrt{x-\xi}}. \tag{65}$$

8.2 A perfect boundary condition

In order to derive a perfect boundary condition for u, the solution in the water, we use the interface conditions (64) in (65). The result is

$$\frac{\partial u(x,0)}{\partial y} = \mathrm{i}\mathcal{M}\Omega \int_0^x \frac{\partial}{\partial \xi}\left\{u(\xi,0)\,\mathrm{e}^{-\mathrm{i}\kappa(x-\xi)}\right\}\frac{d\xi}{\sqrt{x-\xi}}, \tag{66}$$

where $\mathcal{M} = \epsilon\Lambda/(s + \mathrm{i}f)$ and $\Lambda = \Omega_1/\Omega$. This is the exact boundary condition to be imposed on $u(x, y)$ at the porous wall $y = 0$.

The exact condition (66) can be discretized exactly as done previously [50]. For simplicity, we use a uniform discretization in x, with stations at $x_m = m\Delta x$, and then approximate $u(\xi, 0)\,\mathrm{e}^{\mathrm{i}\kappa\xi}$ (rather than just u) by a continuous piecewise-linear function; hence

$$(d/d\xi)\left\{u(\xi,0)\,\mathrm{e}^{\mathrm{i}\kappa\xi}\right\} = \left(u_b^{l+1}\,\mathrm{e}^{\mathrm{i}\kappa x_{l+1}} - u_b^l\,\mathrm{e}^{\mathrm{i}\kappa x_l}\right)/(\Delta x) \quad \text{for } x_l < \xi < x_{l+1}.$$

Then, the integration in (66) can be done analytically; this gives

$$\frac{\partial u_b^j}{\partial y} + \mathcal{M}a u_b^j = \mathcal{M}\sum_{l=0}^{j-1} b_l^j\,\mathrm{e}^{-\mathrm{i}\kappa(j-l)\,\Delta x}\,u_b^l \tag{67}$$

as our generalized impedance boundary condition on the interface $y = 0$, where $u_b^j = u(x_j, 0)$, and the coefficients a and b_l^j are exactly as for the non-porous case (they are defined by equations (54)–(56)).

In the special case where Q_+ is filled with water, we have

$$s = \epsilon = 1 \text{ and } f = 0,$$

whence

$$K_1 = k_0,\ \kappa = 0 \text{ and } \mathcal{M} = 1.$$

Then, (66) and (67) reduce to (51) and (53), respectively.

Note that the formulae (66) and (67) correct those given in [59], where incorrect interface conditions were used.

9 Application of conformal mapping

Wave prediction in realistic coastal situations is often complicated by the layout of breakwaters and other hard structures coupled with variable depths and currents. These complicated situations can often be simplified if a coordinate transformation is used that conforms to the physical boundaries. Hence, we consider a general class of conformal transformations from the Cartesian coordinates (x, y) into boundary-fitted coordinates (u, v), so that no-flow boundary conditions can be applied on coordinate lines. We then describe parabolic approximations in the transformed domain.

Boundary-fitted coordinates have been used extensively in other fields with good success [60], [61]. In the field of wave propagation, Liu & Boissevain [62] transformed the parabolic model into a non-orthogonal coordinate system to examine the propagation of waves in a diverging channel (harbour entrance). Kirby [63] showed that it is important to determine the parabolic model within the mapped domain (rather than apply a conformal transformation to a parabolic model). Tsay *et al.* [64] developed some low-order parabolic approximations for several geometries, while Kirby *et al.* [65] developed parabolic models for several geometric domains for both small- and large-angle parabolic approximations. They also presented laboratory results for the case of the diverging breakwater.

We start with the Helmholtz equation,

$$\nabla^2\psi + [k(\boldsymbol{x})]^2\psi = 0.$$

In the physical domain, $\psi(\boldsymbol{x})$ is found by solving this equation in the given complicated geometry. Alternatively, we can map the problem into a conformal domain, which is identified with the independent variables, $u(x, y)$ and $v(x, y)$. (We will not use $u(\boldsymbol{x})$ as an amplitude function in this section.) The dependent variable becomes $\psi(u, v)$. The mapping procedure is straightforward [65], [66]. For all cases, we arrange that the channel sidewalls are mapped into $v = \pm v_b$.

The resulting governing equation in the conformal domain is similar to that in Cartesian coordinates,

$$\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2} + [K(u, v)]^2\psi = 0, \tag{68}$$

where

$$K^2 = k^2\,J$$

and J is the Jacobian of the transformation, defined by

$$J(u, v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Various numerical models can be developed from (68); see [66] for several, including angular spectrum models [67].

We can only obtain separated solutions of (68) if K^2 is of the form

$$K^2(u, v) = \mathcal{J}_1(u) + \mathcal{J}_2(v).$$

Then, with $\psi(u,v) = U(u)V(v)$, we obtain the following equations for U and V:

$$\begin{aligned} U'' + (\mathcal{J}_1 - \lambda^2)U &= 0, \\ V'' + (\mathcal{J}_2 + \lambda^2)V &= 0. \end{aligned}$$

In particular, if $\mathcal{J}_2 \equiv 0$, the lateral eigenmodes for the channel are

$$V_n(v) = \cos\left[\lambda_n(v + v_b)\right] \qquad \text{with} \qquad \lambda_n = \tfrac{1}{2}n\pi/v_b,$$

just as for the Cartesian case. Alternatively, if $\mathcal{J}_1 \equiv 0$, then we obtain $U(u) = \mathrm{e}^{\mathrm{i}\lambda u}$ as a propagating mode; here, we have replaced λ^2 by $-\lambda^2$, giving

$$V'' + (\mathcal{J}_2 - \lambda^2)V = 0$$

as the equation for the lateral modes.

9.1 Examples

A logarithmic conformal mapping is convenient for illustrating the various approaches to wave modelling. This mapping converts radial lines and circles about the origin in the physical domain into orthogonal straight lines in the mapped domain.

9.1.1 The diverging channel

The first example is a constant depth, radially diverging channel with straight vertical impermeable sidewalls. The mapping is $w = \log(z/r_0)$, where $w = u + \mathrm{i}v$, $z = x + \mathrm{i}y$ and r_0 is the distance from the origin to the mouth of the channel. The mapping can be rewritten as $u = \log(r/r_0)$ and $v = \theta$, which, with the exception of the presence of the logarithm, looks like a polar-coordinate transformation. The channel sidewalls lie on $v = \pm v_b = \pm\theta_\ell$. In terms of x and y, the inverse mapping gives $z = r_0\mathrm{e}^w$, or, $x = r_0\,\mathrm{e}^u\cos v$ and $y = r_0\,\mathrm{e}^u\sin v$. In the z-plane, the waves are supposed to propagate in the positive x-direction, while in the mapped domain, the waves will travel primarily in the positive u-direction.

The Jacobian of the transformation is $J = r_0^2\,\mathrm{e}^{2u}$, which is a function of u only. Thus, $\mathcal{J}_2 \equiv 0$, whence $V(v) = \cos[\lambda(v + v_b)]$ and

$$U'' + [(kr_0\,\mathrm{e}^u)^2 - \lambda^2]U = 0,$$

which has general solution

$$U(u) = AJ_\lambda(kr_0\,\mathrm{e}^u) + BY_\lambda(kr_0\,\mathrm{e}^u),$$

where J_λ and Y_λ are Bessel functions. For rigid walls at $v = \theta = \pm\theta_\ell$ (so that $v_b = \theta_\ell$), and for waves propagating in the direction of u increasing, we readily obtain the solution

$$\psi(r,\theta) = \sum_{n=0}^{\infty} a_n H_{\beta_n}^{(1)}(kr)\cos\beta_n(\theta + \theta_\ell), \tag{69}$$

where $H_\lambda^{(1)} = J_\lambda + \mathrm{i}Y_\lambda$ and $\lambda = \beta_n$, with $\beta_n = \frac{1}{2}n\pi/\theta_\ell$. Given the potential at $r = r_0$, the modal amplitudes a_n are easily obtained.

We note that (69) is the exact linear solution; it can also be obtained by separation of variables in plane polar coordinates [65].

9.1.2 The circular channel

The second example is a constant depth channel with vertical sidewalls laid out in a circular planform. Let r_1 and r_2 be the inner and outer radius of the channel, respectively. The waves are supposed to propagate primarily counter-clockwise in the azimuthal (θ) direction, from the mouth of the channel located at $\theta = -\pi/2$. In the mapped domain, the channel is straight, with the waves again propagating in the positive u-direction. Here the conformal map is somewhat different (to keep the same u principal propagation directions): $w = \pi/2 - \mathrm{i}\log(z/r_\mathrm{m})$, where $r_\mathrm{m} = \sqrt{r_1 r_2}$. This corresponds to $u = \pi/2 + \theta$ and $v = \log(r_\mathrm{m}/r)$. The outer sidewall of the channel is mapped to $v = -v_b = \log(r_\mathrm{m}/r_2) = -\frac{1}{2}\log(r_2/r_1)$, while the inner wall is mapped to $v = v_b$. In terms of z, we have $z = r_\mathrm{m}\mathrm{e}^{\mathrm{i}(w-\pi/2)}$, which leads to $x = r_\mathrm{m}\mathrm{e}^{-v}\sin u$ and $y = -r_\mathrm{m}\mathrm{e}^{-v}\cos u$.

The Jacobian of this transformation is $J = r_\mathrm{m}^2\mathrm{e}^{-2v}$, which is a function of v only. Thus, $\mathcal{J}_1 \equiv 0$, whence $U(u) = \mathrm{e}^{\mathrm{i}\lambda u}$ for propagation in the direction of u increasing. $V(v)$ satisfies

$$V'' + [(kr_\mathrm{m}\mathrm{e}^{-v})^2 - \lambda^2]V = 0, \tag{70}$$

which has general solution

$$V(v) = AJ_\lambda(kr_\mathrm{m}\mathrm{e}^{-v}) + BY_\lambda(kr_\mathrm{m}\mathrm{e}^{-v}).$$

At the outer wall $r = r_2$, we have $v = \frac{1}{2}\log(r_1/r_2) = -v_b$ and the boundary condition $V'(-v_b) = 0$; therefore

$$V(v) = Y_\lambda'(kr_2)J_\lambda(kr_\mathrm{m}\mathrm{e}^{-v}) - J_\lambda'(kr_2)Y_\lambda(kr_\mathrm{m}\mathrm{e}^{-v}).$$

At the inner wall $r = r_1 < r_2$, we have $v = v_b$ and $V'(v_b) = 0$, giving

$$Y'_\lambda(kr_1)J'_\lambda(kr_2) - J'_\lambda(kr_1)Y'_\lambda(kr_2) = 0. \tag{71}$$

This is an equation for λ. It is known that (71) has discrete roots; call them $\lambda = \alpha_n$, with $n = 0, 1, 2, \ldots$. There is only a finite number of real roots ($0 < \alpha_n < kr_2$); these give the propagating modes. Equation (71) also has an infinite number of purely imaginary solutions; those with positive imaginary parts give the evanescent modes. These solutions have been discussed by Buchholz [68] in the context of curved electromagnetic wave guides, by Johns & Hamzah [69] in the context of long water waves, and by Rostafinski [70] in the context of acoustics.

Ordering the real eigenvalues from the largest to the smallest, we find that the first eigenvalue corresponds to the zero-th mode, which has no zero crossing in the transverse (radial) direction. Therefore the mode looks like a propagating wave train, but confined to the outer wall; it is the annular equivalent of the 'whispering gallery mode' as it is large on the outer radius and decays rapidly and monotonically in the (negative) r-direction. The next eigenvalue corresponds to the first mode, with one zero crossing, and so on.

The problem of solving (70), together with $V'(\pm v_b) = 0$, is a Sturm–Liouville problem. Let $V_n(v)$ be a solution corresponding to the eigenvalue $\lambda = \alpha_n$,

$$V_n(v) = A_n \left\{ Y'_{\alpha_n}(kr_2)J_{\alpha_n}(kr_\mathrm{m}\mathrm{e}^{-v}) - J'_{\alpha_n}(kr_2)Y_{\alpha_n}(kr_\mathrm{m}\mathrm{e}^{-v}) \right\}$$

(recall that $r = r_\mathrm{m}\mathrm{e}^{-v}$). These eigenfunctions are orthogonal; moreover, they can be made orthonormal by an appropriate choice of the constant A_n. They are also complete, so that we have

$$\psi(u, v) = \sum_{n=0}^{\infty} a_n \, \mathrm{e}^{\mathrm{i}\alpha_n u} \, V_n(v).$$

At the beginning of the channel, $u = 0$ ($\theta = -\pi/2$), we know $\psi(0, v)$, whence the coefficients a_n can be found:

$$a_n = \int_{-v_b}^{v_b} \psi(0, v) \, V_n(v) \, dv.$$

Again, this solution is exact; it can also be obtained by separation of variables in plane polar coordinates [65]. Numerical evaluations of this solution (for waves incident into a 180° turn) are given in [65] and [66].

9.2 Parabolic models

In the mapped domain, we have to solve (68). This is exactly the equation discussed in section 3, and so we can choose any appropriate parabolic model. Assume that waves propagate mainly in the positive u-direction. Then, one choice is to write

$$\psi(u, v) = A(u, v) \exp\left\{ \mathrm{i} \int^u K_1(u') \, du' \right\},$$

where $K_1(u) = K(u, v_0)$ is a 'reference phase function' based on one particular value of v. The parabolic equation for the amplitude A is (17):

$$2\mathrm{i}K_1 \frac{\partial A}{\partial u} + \mathrm{i}K'_1 A + (K^2 - K_1^2)A + \frac{\partial^2 A}{\partial v^2} = 0.$$

This is equation (28) in [65]. Alternatively, if we can identify a representative constant wavenumber, K_0, then we can write

$$\psi(u, v) = A(u, v) \, \mathrm{e}^{\mathrm{i}K_0 u}.$$

This leads to various possible equations for A. For example, the equation of Corones and Radder, namely (23), becomes

$$2\mathrm{i}K \frac{\partial A}{\partial u} + 2K(K - K_0)A + \mathrm{i}A \frac{\partial K}{\partial u} + \frac{\partial^2 A}{\partial v^2} = 0.$$

This is (30) in [65]. This paper describes further parabolic models, and gives comparisons between numerical solutions, the exact solutions described above, and some experiments.

10 Weakly nonlinear waves in shallow water

To date, most of the emphasis in parabolic model development has centred on applications of the mild-slope equation to problems of intermediate-depth wave propagation. These approximations may be extended to include nonlinear effects by utilizing the correspondence between the parabolic amplitude evolution equation and the cubic Schrödinger equation for narrow banded wave trains [11, 12]. However, the third-order Stokes expansion on which these models are based is not valid in the

limit of shallow water. It is possible to empirically modify the model coefficients to avoid the shallow water singularity [71], but the fact remains that the narrow-banded envelope equation is not a good representation of physics in the final stages of wave shoaling or during wave breaking, where the wave form and spectral content evolve rapidly due to near-resonant interactions at the second order in nonlinearity.

The construction of a more valid basis for shallow water wave propagation rests on a recognition of the fact that wave propagation velocities approach a common value $\sqrt{gh}$ independent of frequency, thus allowing all waves to be regarded as non-dispersive shallow water waves with only weak deviations in phase speed. If we let $\mu = kh$ characterize dispersiveness and $\delta = a/h$ characterize nonlinearity, then the scaling regime which provides most of the basis for modern work on propagation modelling is the Boussinesq regime $\mu \ll 1$, $\delta \ll 1$, $\delta/\mu^2 = O(1)$, which encompasses the Boussinesq equations, the one-dimensional Korteweg–deVries equation, and the weakly two-dimensional Kadomtsev–Petviashvili equation. The foundation of this theory is introduced in a separate chapter [72]. We concentrate here on the aspects of the theory related to the construction of parabolic approximations.

10.1 Boussinesq equations

The Boussinesq equations are derived by constructing a series solution to Laplace's equation in the fluid interior, and then using the resulting series to specify the velocity potential appearing in the free surface boundary conditions [72]. The order of approximation in the resulting evolution equations corresponds to the level of truncation in the expansion parameters μ and δ; the Boussinesq equations are obtained by retaining only the leading order effects of each. For variable depth, Boussinesq equations were developed by Peregrine [73] using the depth-averaged horizontal velocity and the surface displacement as dependent variables. Let a_0, h_0 and ω be the characteristic amplitude, water depth and frequency, respectively, of the wave motion. These are used to define dimensionless variables as follows:

$$t' = \omega t, \quad (x', y') = \frac{\omega}{\sqrt{gh_0}}(x, y), \quad z' = \frac{z}{h_0},$$

$$h' = \frac{h}{h_0}, \quad \overline{\boldsymbol{u}}' = \frac{h_0}{a_0\sqrt{gh_0}}\overline{\boldsymbol{u}}, \quad \eta' = \frac{\eta}{a_0};$$

here, $\overline{\boldsymbol{u}}$ represents the depth-averaged horizontal velocity vector. There are two dimensionless parameters, $\delta = a_0/h_0$ and $\mu^2 = \omega^2 h_0/g$. Assuming that these are of the same (small) size, and omitting the primes, the following Boussinesq equations are obtained,

$$\frac{\partial \eta}{\partial t} + \nabla \cdot [(h + \delta\eta)\overline{\boldsymbol{u}}] = 0, \tag{72}$$

$$\frac{\partial \overline{\boldsymbol{u}}}{\partial t} + \delta(\overline{\boldsymbol{u}} \cdot \nabla)\overline{\boldsymbol{u}} + \nabla\eta = \mu^2 \left\{ \frac{h}{2}\frac{\partial}{\partial t}\nabla[\nabla \cdot (h\overline{\boldsymbol{u}})] - \frac{h^2}{6}\frac{\partial}{\partial t}\nabla(\nabla \cdot \overline{\boldsymbol{u}}) \right\}, \tag{73}$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$ is the horizontal gradient operator. The resulting model equations are rotationally symmetric in the horizontal (x, y) plane, and thus are fully two-dimensional in the same sense as the Helmholtz or mild-slope equations studied above. The evolution equations differ from the models above in that the entire frequency spectrum would be represented by a single calculation, rather than having each frequency component represented by a separate elliptic model equation. This distinction is lost during the construction of the parabolic approximation, where the equations are transformed from the time domain into the frequency domain. Various authors have extended the range of application of the Boussinesq model by making adjustments to dispersive and nonlinear terms. This work is reviewed in the present volume by Kirby [72].

10.2 Kadomtsev–Petviashvili equation: a parabolic time-domain model

Before constructing a parabolic approximation directly from the Boussinesq equations, we review what is essentially a parabolic form of the time-dependent evolution equations themselves. This approximation was first introduced by Kadomtsev & Petviashvili [74], and the resulting class of equations are generically referred to as KP equations. The general development of these equations in the context of water wave theory is discussed in [72].

The parabolic approximations developed above imply a relationship between x and y direction wavenumber coefficients $\ell = k\cos\theta$ and $m = k\sin\theta$. Referring to §3.4, the exact relationship between k, ℓ and m is given by

$$k = \sqrt{\ell^2 + m^2} = \ell\sqrt{1 + m^2/\ell^2} \tag{74}$$

With θ small, the ratio appearing in the last expression in (74) is small, and the square root may be accurately approximated by the binomial expansion, giving

$$k = \ell\left(1 + \frac{1}{2}\frac{m^2}{\ell^2}\right). \tag{75}$$

Let us compare this to the representation of a plane wave in the KP equation. The KP equation for a dependent variable η in a uniform medium may be written as

$$\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial t} + c\frac{\partial \eta}{\partial x} + \frac{3c}{2h}\eta\frac{\partial \eta}{\partial x} + \frac{ch^2}{6}\frac{\partial^3 \eta}{\partial x^3}\right) + \frac{c}{2}\frac{\partial^2 \eta}{\partial y^2} = 0. \tag{76}$$

Assuming that η is written as

$$\eta = a\,\mathrm{e}^{\mathrm{i}(\ell x + my - \omega t)} \tag{77}$$

and substituting in the linearized, nondispersive version of (76) then gives

$$\frac{\omega}{c} = k = \ell\left(1 + \frac{1}{2}\frac{m^2}{\ell^2}\right), \tag{78}$$

which is identical to (75). The KP equation thus employs exactly the same assumptions about angular relations as are used in constructing the lowest order parabolic approximations for plane wave propagation. Interestingly, there have been no attempts (that we know of) at extending the KP equation formulation to include higher-order approximations such as (29). It is also not clear that doing so would yield a model equation whose numerical solution would be more efficient to obtain than the solution of the fully two-dimensional Boussinesq equations.

Liu *et al.* [75] have used a variable-depth form of the KP equation to develop a parabolic equation system for shallow water wave propagation. The resulting equations are essentially equivalent to the equations derived directly from the Boussinesq equations, illustrated in the next section.

10.3 Parabolic approximation of Boussinesq equations

Liu *et al.* [75] examined the propagation of periodic waves in shallow water, using the (dimensionless) Boussinesq equations, equations (72)

and (73) as well as an equivalent formulation based on a variable depth KP equation. They assumed a Fourier expansion for η and $\overline{\boldsymbol{u}}$ in time,

$$\begin{aligned}\eta(\boldsymbol{x},t) &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\zeta_n(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}nt},\\ \overline{\boldsymbol{u}}\,(\boldsymbol{x},t) &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\overline{\boldsymbol{u}}_n(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}nt},\end{aligned}$$

where ζ_{-n} and $\overline{\boldsymbol{u}}_{-n}$ are the complex conjugates of ζ_n and $\overline{\boldsymbol{u}}_n$, respectively. Substituting these into the Boussinesq equations yields an equation for each Fourier mode:

$$-\mathrm{i}n\zeta_n + \nabla\cdot(h\overline{\boldsymbol{u}}_n) + \frac{\delta}{2}\sum_{s=-\infty}^{\infty}\nabla\cdot(\zeta_s\overline{\boldsymbol{u}}_{n-s}) = 0, \tag{79}$$

$$-\mathrm{i}n\overline{\boldsymbol{u}}_n + \left(1 - \frac{\mu^2 n^2}{3}h\right)\nabla\zeta_n + \frac{\delta}{4}\sum_{s=-\infty}^{\infty}\nabla\left(\overline{\boldsymbol{u}}_s\cdot\overline{\boldsymbol{u}}_{n-s}\right) = 0. \tag{80}$$

At lowest order,

$$\overline{\boldsymbol{u}}_n = -\frac{\mathrm{i}}{n}\nabla\zeta_n\left(1 + O(\delta,\mu^2)\right) \tag{81}$$

$$\nabla\cdot\overline{\boldsymbol{u}}_n = \frac{\mathrm{i}n}{h}\zeta_n\left(1 + O(\delta,\mu^2)\right) \tag{82}$$

except for $n = 0$, which gives

$$\begin{aligned}\overline{\boldsymbol{u}}_0 &= -\frac{\delta}{2h}\sum_s \zeta_s\overline{\boldsymbol{u}}_{-s} + O\left(\delta^2,\mu^4,\delta\mu^2\right),\\ \zeta_0 &= -\frac{\delta}{4}\sum_s \overline{\boldsymbol{u}}_s\cdot\overline{\boldsymbol{u}}_s + O\left(\delta^2,\mu^4,\delta\mu^2\right).\end{aligned}$$

Substituting equations (81) and (82) into equations (79) and (80) and eliminating $\overline{\boldsymbol{u}}_n$ gives a set of elliptic equations for ζ_n,

$$\begin{aligned}&\nabla\cdot\left[\left(h - \frac{\mu^2 n^2 h^2}{3}\right)\nabla\zeta_n\right] + n^2\zeta_n\\ &= \frac{\delta}{2h}\left\{\sum_s (n^2 - s^2)\,\zeta_s\zeta_{n-s} - h\sum_{s\neq n}\left(\frac{n+s}{n-s}\right)\nabla\zeta_s\cdot\nabla\zeta_{n-s}\right.\\ &\quad\left. - 2h^2\sum_{s\neq 0,n}\frac{1}{s(n-s)}\left(\frac{\partial^2\zeta_s}{\partial x^2}\frac{\partial^2\zeta_{n-s}}{\partial y^2} - \frac{\partial^2\zeta_s}{\partial x\,\partial y}\frac{\partial^2\zeta_{n-s}}{\partial x\,\partial y}\right)\right\},\end{aligned} \tag{83}$$

for each n, with an error of $O(\delta^2, \delta\mu^2, \mu^4)$.

To obtain a parabolic form, Liu *et al.* [75] assumed that the free surface displacement ζ_n could be written as

$$\zeta_n = \psi_n \, \mathrm{e}^{\mathrm{i}nx},$$

where $\psi_n(\boldsymbol{x})$ is a spatially varying amplitude. Substituting into (83) and assuming that the x-variation of ψ_n is small, a parabolic equation results for each amplitude:

$$\begin{aligned} 2\mathrm{i}n\frac{\partial\psi_n}{\partial x} \;+\; & \frac{\partial^2\psi_n}{\partial y^2} + \frac{1}{G_n}\frac{\partial G_n}{\partial y}\frac{\partial\psi_n}{\partial y} + \left[\frac{\mathrm{i}n}{G_n}\frac{\partial G_n}{\partial x} - n^2\left(1 - \frac{1}{G_n}\right)\right]\psi_n \\ = \; & \frac{\delta}{2hG_n}\left\{\sum_{s=-\infty}^{\infty}\left[hs(n+s) + n^2 - s^2\right]\psi_s\psi_{n-s}\right. \\ & - h\sum_{s\neq n}\left(\frac{n+s}{n-s}\right)\frac{\partial\psi_s}{\partial y}\frac{\partial\psi_{n-s}}{\partial y} \\ & \left. + 2h^2\sum_{s\neq n}\frac{1}{n-s}\left[s\psi_s\frac{\partial^2\psi_{n-s}}{\partial y^2} - (n-s)\frac{\partial\psi_s}{\partial y}\frac{\partial\psi_{n-s}}{\partial y}\right]\right\}, \end{aligned}$$

where

$$G_n = h - \tfrac{1}{3}\mu^2 n^2 h^2.$$

This parabolic form of the Boussinesq equations is also solved conveniently by the Crank–Nicolson scheme; see §4.

Liu *et al.* [75] examined the refraction and shoaling of a shallow water wave over a topographic lens, using the laboratory study of Whalin [76]. They compared the amplitudes of the first three harmonics down the centreline of the wave channel with laboratory data from three different water depths; they found best agreement with the shallowest case.

Yoon & Liu [77] applied a similar model to describe Mach stem reflection along a breakwater, while Yoon & Liu [78] consider the case where a strong current field is imposed and interacts with the propagating wave components. Chen & Liu [79] have extended the parabolic formulation to incorporate improved Boussinesq model equations, and Kaihatu & Kirby [80] have further modified the parabolic model formulation to improve the accuracy of shoaling terms and to incorporate dispersive effects in nonlinear terms. Finally, Kaihatu & Kirby [81] have developed a parabolic model where wave coupling takes place through

quadratic three-wave interactions, as in the shallow water models developed here, but where correct linear dispersion effects are incorporated at all resolved frequencies.

Liu [82] has described a model for energy dissipation in breaking waves in the parabolized Boussinesq model. Further applications are reviewed in [72].

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