BRIEF COMMUNICATIONS

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Propagation of surface waves over an undulating bed

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Recent results of Davies *et al.* [Phys. Fluids A **1**, 1331 (1989)], which cast the problem of scattering of long surface waves by sinusoidal bed undulations into a Mathieu equation, are extended here to include the case of dispersive, intermediate depth waves. The present formulation is restricted to linear monochromatic wave motions and the bed undulation amplitude is assumed to be small relative to the total water depth.

The problem of reflection of surface water waves by undulating bottom forms has drawn considerable attention in recent years because of its possible importance in the context of coastal geomorphology. In a recent paper, Davies *et al.*[1] have considered the case of a sinusoidal bed undulation of small amplitude superimposed on a region of otherwise constant depth, and have shown that the wave field is governed (to first order in a small parameter based on bar amplitude) by the Mathieu equation. The analysis is restricted to nondispersive, linear, monochromatic long waves. Davies *et al.* also showed that, in the case of nonresonant reflection, a subsequent expansion assuming a reflected wave of $O(\epsilon = \text{bar amplitude/water depth} \ll 1)$ and transmitted wave of $O(1)$ recovers the long-wave limit of the reflection coefficient found by Davies and Heathershaw.[2] Close to the Bragg resonance condition, a rescaling is necessary, with both incident and reflected waves taken to be $O(1)$ and the frequency detuning away from resonance to be $O(\epsilon)$. Analysis of this case recovers the long-wave limit of the results of Mei.[3]

Here, we point out that the analysis given by Davies *et al.* may be extended simply to intermediate depth, dispersive waves, yielding the Mathieu equation formulation with altered coefficients. Thus the limitation to nondispersive long waves is alleviated. We also consider the case of oblique incidence on the bar field.

For the case of waves in intermediate water depth $[kh = O(1)]$, we may regard the depth $h(x,y)$ to be composed of a slowly varying mean component $\overline{h}(x,y)$ and a rapid superposed undulation $\delta(x,y)$, according to

$$h = \overline{h}(x,y) - \delta(x,y). \tag{1}$$

We assume the scaling restrictions

$$|\nabla\overline{h}/\overline{k}\overline{h}| = O(\epsilon), \quad |\delta/\overline{h}| = O(\epsilon), \quad \epsilon \ll 1. \tag{2}$$

Under these conditions, Kirby[4] showed that the surface displacement $\eta(x,y)$ of a time-periodic wave of frequency ω is governed by an extended mild-slope equation, given by

$$\nabla\cdot(\overline{C}\overline{C}_g \nabla\eta) + \overline{k}^2\overline{C}\overline{C}_g\,\eta - (g/\cosh^2 \overline{k}\overline{h})\nabla\cdot(\delta\,\nabla\eta) = 0, \tag{3}$$

correct to $O(\epsilon)$. Here,

$$\omega^2 = g\overline{k}\tanh \overline{k}\overline{h},$$
$$\overline{C} = \frac{\omega}{\overline{k}}, \quad \overline{C}_g = \frac{\partial\omega}{\partial\overline{k}}. \tag{4}$$

Since derivatives of $\cosh \overline{k}\overline{h}$ are of $O(\epsilon)$, we may write to the same level of approximation[5]

$$\nabla\cdot(f\nabla\eta) + \overline{k}^2 p\eta = 0, \tag{5}$$

where

$$p = \overline{C}\overline{C}_g, \quad f = p - g\delta/\cosh^2 \overline{k}\overline{h}. \tag{6}$$

Employing the change of variable

$$\eta = f^{-1/2}W \tag{7}$$

changes (5) to the form

$$\nabla^2 W + [\overline{k}^2 + A(\overline{k}^2\delta + \nabla^2\delta/2)]W = 0, \tag{8}$$

where

$$A = g/\overline{C}\overline{C}_g \cosh^2 \overline{k}\overline{h} \tag{9}$$

and where terms proportional to the (slope)^2 and curvature of $\overline{h}$ have been neglected as being of $O(\epsilon^2)$. (These terms must be retained in the vicinity of shorelines.) Note that as $\overline{k}\overline{h} \to 0$, $A \to [1 + O(\overline{k}\overline{h})^2]/\overline{h}$.

We restrict our attention to the case $\overline{h} = \overline{h}(x)$, $\delta = \delta(x)$, and $\partial/\partial y \equiv 0$, where (x,y) is the horizontal plane. Equation (8) reduces to

$$W_{xx} + [\overline{k}^2 + A(\overline{k}^2\delta + \delta_{xx}/2)]W = 0. \tag{10}$$

Consider the case $\overline{h}$ = const and

$$\delta(x) = -\epsilon\overline{h}\cos lx, \tag{11}$$

which corresponds to Davies *et al.* Here, $\epsilon = b/\overline{h}$, b is the bar amplitude, and l is the bar wavenumber. Using the change of coordinates $lx = 2z$, we obtain

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$$W_{zz} + \lambda^2\{1 - \epsilon A\bar{h}\,[(\lambda^2 - 2)/\lambda^2]\cos 2z\} W = 0, \quad (12)$$

where $\lambda^2 = (2\bar{k}/l)^2$. As $\bar{k}\bar{h} \to 0$, $A\bar{h} \to 1 + O(\bar{k}\bar{h})^2$ and we recover the model given by Davies *et al.* Equation (12) is the Mathieu equation

$$W_{zz} + (a - 2q\cos 2z) W = 0, \quad (13)$$

with

$$a = \lambda^2, \quad q = \epsilon A\bar{h}(\lambda^2 - 2)/2. \quad (14)$$

The revision of the parameter q in Davies *et al.* by the form given in (14) then allows for the recovery of the general results of Davies and Heathershaw[2] and Mei,[3] rather than just the long-wave limit.

For the case of a bar field of finite extent, matching conditions between the solution over the bar field and the solutions in the uniform domains to either side are required. We take $\delta(z) \neq 0$ in the interval $-\pi/4 < z < (N - \frac{1}{4})\pi$, where N is the number of full bar wavelengths and the shift of 45° causes the undulation to have a value of zero at the bar field edges. Equation (12) may then be solved in the finite domain with the use of appropriate boundary conditions. In $z < -\pi/4$, W may be written as

$$W = W_I + W_R = e^{i\lambda z} + \mathrm{R}e^{-i\lambda z}, \quad z < -\pi/4, \quad (15)$$

where R is a complex reflection coefficient. In $z > (N - \frac{1}{4})\pi$, we have

$$W = W_T = Te^{i\lambda z}, \quad z > (N - \tfrac{1}{4})\pi, \quad (16)$$

where T is the complex transmission coefficient. Noting that the reflected wave W_R and the transmitted wave W_T should satisfy appropriate radiation conditions in their respective domains, we obtain the mixed boundary conditions on W,

$$W_z = i\lambda(2W_I - W), \quad z = -\pi/4, \quad (17)$$

$$W_z = i\lambda W, \quad z = (N - \tfrac{1}{4})\pi. \quad (18)$$

Equations (17) and (18), together with (13), may be used as a basis for direct numerical computations. Kirby[4] has compared a direct solution of (3) with the experimental data of Davies and Heathershaw.[2] Solutions of (13) are not substantially different and show good agreement with experimental data.

We further show the extension to the case of oblique incidence on the one-dimensional bar field. Returning to the dispersive wave model, we obtain

$$\hat{W}_{xx} + [(\bar{k}^2 - m^2) + A(\bar{k}^2\delta - \delta_{xx}/2)]\,\hat{W} = 0,$$
$$W = \hat{W}e^{imy}, \quad (19)$$

where $\delta = \delta(x)$ only, and

$$m = \bar{k}\sin\theta = \text{const.} \quad (20)$$

The transformations presented above give the Mathieu equation (13) with q given by (14) and

$$a = \lambda^2 - \gamma^2, \quad \gamma = 2m/l = (2\bar{k}/l)\sin\theta. \quad (21)$$

Then

$$a = [(2\bar{k}/l)\cos\theta]^2 > 0 \quad (22)$$

always, and the basic form of the equation is unchanged. Explicit results near resonance for this case have recently been given elsewhere.[6]

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