JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 89, NO. C1, PAGES 745–747, JANUARY 20, 1984

# A Note on Linear Surface Wave-Current Interaction Over Slowly Varying Topography

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A mild slope wave equation is derived which governs the propagation of linear surface waves in the presence of large ambient currents. The equation is shown to differ from two previously derived models, and arguments for the validity of the new version in comparison to previous versions are presented. A linearized evolution equation and parabolic equation approximation are constructed in order to show the correspondence between the present corrected version and a previously derived version of the time-dependent Schrödinger equation.

## 1. INTRODUCTION

Recently, two studies have provided different versions of a hyperbolic wave equation governing the propagation of waves in the presence of varying depth and currents in the mild slope approximation. The first derivation, given by *Booij* [1981], resulted in an equation given by

$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla_h\right)\left(\frac{\partial \Phi}{\partial t} + \nabla_h\cdot(\mathbf{U}\Phi)\right) - \nabla_h\cdot(CC_g\nabla_h\Phi) + (\sigma^2 - k^2CC_g)\Phi = 0 \quad (1)$$

Here, $\mathbf{U}(x, y)$ is the ambient current, $\nabla_h$ represents a horizontal gradient operator, and $\Phi(x, y, t)$ is the surface potential (at $z = 0$, the mean water level) of the linearized wave motion. The remaining coefficients are defined according to

$$\sigma^2 = gk \tanh kh \qquad k = |\mathbf{k}| \quad (2)$$

$$\omega = \sigma + \mathbf{k}\cdot\mathbf{U} = \text{absolute frequency} \quad (3)$$

$$C = \sigma/k \quad (4)$$

$$C_g = \partial\sigma/\partial k \quad (5)$$

Equation (1) may be rearranged slightly into the form

$$\frac{D^2\Phi}{Dt^2} + (\nabla_h\cdot\mathbf{U})\frac{D\Phi}{Dt} + \left\{\frac{D}{Dt}(\nabla_h\cdot\mathbf{U})\right\}\Phi - \nabla_h\cdot(CC_g\nabla_h\Phi) + (\sigma^2 - k^2CC_g)\Phi = 0 \quad (6)$$

where

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla_h\right)$$

It can be shown that the third term in (6) arises in Booij's result due to the use of an incorrect form of the dynamic free surface boundary condition, given by

$$\frac{\partial\Phi}{\partial t} + \nabla_h\cdot(\mathbf{U}\Phi) + g\eta = 0$$

rather than the correct form given below in (16).

*Liu* [1983] has provided a second derivation of the mild slope equation and obtained the result

$$\frac{D^2\Phi}{Dt^2} - \nabla_h\cdot(CC_g\nabla_h\Phi) + (\sigma^2 - k^2\, CC_g)\Phi = 0 \quad (7)$$

which also neglects the second term in (6) due to limitations in the procedure used to derive the equation.

The purpose of the present investigation is to demonstrate that (6), after neglecting the third term, represents the correct mild slope equation for linear wave motion. A derivation of the equation using a Lagrangian formulation is provided in section 2. Then, in section 3, it is shown that the conservation equation for wave action follows directly from the correct form of the wave equation without further assumption, and the point of difficulty in obtaining this relation from (7) is illustrated. Finally, in section 4, a parabolic approximation is derived and consistancy with a previously derived, time-dependent Schrödinger equation for one-directional wave propagation is shown.

## 2. THE MILD SLOPE WAVE EQUATION

Two scale parameters are established in order to facilitate bookkeeping. The first is given by the Stokes wave steepness parameter $\varepsilon = O(kA)$, where $A$ is the wave amplitude. A second parameter $\delta = O(\nabla_h h/kh)$ is related to the rate of change of depth over the space of a wavelength. The velocity potential $\phi(x, y, z, t)$ and free surface $\eta(x, y, t)$ appropriate to the assumption of $\varepsilon$, $\delta \ll 1$ and $\delta = O(\varepsilon)$ may be obtained through the expansion procedure of *Chu and Mei* [1970] after allowing for currents of $O(1)$ and are given by

$$\phi = \delta^{-1}\phi_0(\delta x, \delta y, \delta t) + \varepsilon\phi_1(x, y, z, t) + O(\varepsilon^2) \quad (8a)$$

$$\eta = \eta_0(\delta x, \delta y, \delta t) + \varepsilon\eta_1(x, y, t) + O(\varepsilon^2) \quad (8b)$$

Here, $\phi_0$ is the potential for the $O(1)$ mean current,

$$\mathbf{U}(x, y, t) = \nabla_h\phi_0 \quad (9)$$

and $\phi_1$ is given by

$$\phi_1 = -\frac{ig}{2\sigma} f A e^{i\psi} + \text{complex conjugate} \quad (10)$$

where

$$f(\delta x, \delta y, z) = \frac{\cosh k(h+z)}{\cosh kh} \quad (11)$$

$$\mathbf{k} \equiv \nabla_h\psi \qquad \omega \equiv -\frac{\partial\psi}{\partial t} \quad (12)$$

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Paper number 3C1138.
0148-0227/84/003C-1138$02.00

The potential $\phi$ for a general wave motion in the mild slope approximation can then be written as

$$\phi = \phi_0 + \varepsilon f \Phi(x, y, t) \tag{13}$$

The variational principle governing irrotational fluid motion is given by [*Luke*, 1967]

$$\delta \int_t \int_{\mathbf{x}} L \, d\mathbf{x} \, dt = 0 \qquad L = \int_{-h_0}^{\eta} \left\{ \phi_t + \frac{1}{2} (\nabla \phi)^2 + gz \right\} dz \tag{14}$$

where $h_0$ is the local depth with respect to still water level. The limit of integration $h_0$, $\eta$ may be shifted by defining the local depth with respect to mean water level: $h = h_0 + \eta_0$. The upper limit is shifted to $\varepsilon \eta_1$. In (14), $\nabla$ is a three-dimensional gradient operator. After substituting (8*b*) and (13) into (14), performing the integration over depth, and expanding the resulting expression in Taylor series about the $O(1)$ mean water level, terms of $O(\varepsilon^2)$ are collected to yield the expression

$$L(O(\varepsilon^2)) = \frac{g\eta_1^{\,2}}{2} + \eta_1 \frac{D\Phi}{Dt} + \left(\frac{CC_g}{2g}\right)(\nabla_h \Phi)^2 + (\sigma^2 - k^2 CC_g) \frac{\Phi^2}{2g} + \int_{-h}^{0} \left(\frac{\partial f}{\partial h}\right)^2 (\nabla h)^2 \, dz \, \frac{\Phi^2}{2} + \left\{ \int_{-h}^{0} f \frac{\partial f}{\partial h} \nabla_h h \, dz \cdot \nabla_h \Phi \right\} \Phi \tag{15}$$

Varying $L$ with respect to $\eta_1$ yields the free surface boundary condition

$$g\eta_1 = -\frac{D\Phi}{Dt} \tag{16}$$

Varying $L$ with respect to $\Phi$, performing the partial integration, and invoking the arbitrariness of variations $\delta\Phi$ yields

$$-\frac{\partial \eta_1}{\partial t} - \nabla_h \cdot (\mathbf{U}\eta_1) - \nabla_h \cdot \left(\frac{CC_g}{g} \nabla_h \Phi\right) + \frac{(\sigma^2 - k^2 CC_g)}{g} \Phi = \left\{ -\int_{-h}^{0} \left(\frac{\partial f}{\partial h}\right)^2 (\nabla_h h)^2 \, dz + \frac{\partial}{\partial h} \left[ \int_{-h}^{0} f \frac{\partial f}{\partial h} \, dz \right] (\nabla_h h)^2 \right\} \Phi \tag{17}$$

The right-hand side of (17) is formally of $O(\delta^2)$ and may be dropped. Using (16) to eliminate $\eta_1$ gives

$$\frac{D^2\Phi}{Dt^2} + (\nabla_h \cdot \mathbf{U}) \frac{D\Phi}{Dt} - \nabla_h \cdot (CC_g \nabla_h \Phi) + (\sigma^2 - k^2 CC_g)\Phi = 0 \tag{18}$$

which is the final form of the wave equation. *Booij* [1981] would have obtained (18) if he had used the correct form of the free surface boundary conditions in his derivation of the Lagrangian $L$.

## 3. Conservation of Wave Action

The conservation law for the wave action $\mathscr{A}$, defined according to

$$\mathscr{A} = \tfrac{1}{2} \rho g A^2 / \sigma \tag{19}$$

follows directly from (18) without further approximation. The potential $\Phi$ is written as

$$ig^{-1}\Phi = Re^{i\psi} \qquad R = A/\sigma \tag{20}$$

and $R$ and $\psi$ are taken to be purely real quantities. After substituting (20) into (18), the properties of $R$ and $\psi$ may be used to set real and imaginary parts of the resulting expression equal to zero. The imaginary part of (18) is then given by

$$\frac{\partial \sigma}{\partial t} R + 2\sigma \frac{\partial R}{\partial t} + 2\sigma \mathbf{U} \cdot \nabla_h R + \nabla_h \cdot (\sigma \mathbf{U}) R + \nabla_h \cdot (\mathbf{k} CC_g) R + 2CC_g \mathbf{k} \cdot \nabla_h R = 0 \tag{21}$$

If (7) had been taken as the starting point, the fourth term in (21) would have been replaced by $(\mathbf{U} \cdot \nabla_h \sigma)R$; the following steps would then not lead to the correct result. After multiplying by $R$ and noting that

$$\mathbf{k} CC_g = (kC)\left(C_g \frac{\mathbf{k}}{k}\right) = \sigma \mathbf{C}_g \tag{22}$$

(21) becomes

$$\frac{\partial}{\partial t} (\sigma R^2) + \sigma \mathbf{U} \cdot \nabla_h (R^2) + \sigma \mathbf{C}_g \cdot \nabla_h (R^2) + \nabla_h \cdot (\sigma \mathbf{U}) R^2 + \nabla_h \cdot (\sigma \mathbf{C}_g) R^2 = 0$$

or

$$\frac{\partial}{\partial t} (\sigma R^2) + \nabla_h \cdot (\sigma R^2 (\mathbf{C}_g + \mathbf{U})) = 0 \tag{23}$$

Upon using (19) and (20), the final conservation equation is obtained:

$$\frac{\partial}{\partial t} (\mathscr{A}) + \nabla_h \cdot (\mathscr{A} (\mathbf{C}_g + \mathbf{U})) = 0 \tag{24}$$

This has been shown to be the correct conservation relation for an energylike quantity by numerous investigators, including *Whitham* [1967] and *Bretherton and Garrett* [1968]. Equation (24) does not follow naturally from the wave equation given by (7). The added term in (6) is $O(\delta^2)$, and (24) follows after neglecting it.

## 4. Evolution Equations and the Parabolic Approximation

As a further example, an evolution equation for linear waves in the presence of a current is derived. Here, $\Phi$ is assumed to take on a slightly different form given by

$$ig^{-1}\Phi = Re^{i\psi} \qquad \psi = \int^x k \, dx - \omega t \tag{25}$$

where $x$ is the principal direction of propagation. Since the processes of refraction and diffraction may turn the calculated wave away from the $x$ direction, $R$ (or $A$) must be allowed to be complex in order to absorb the difference between the assumed and actual phase. Substituting (25) into (18) yields a complex form of (21) given by

$$i\left\{\frac{\partial \sigma}{\partial t} R + 2\sigma \frac{\partial R}{\partial t} + 2\sigma \mathbf{U} \cdot \nabla_h R + \nabla_h \cdot (\sigma \mathbf{U}) R + \frac{\partial(\sigma C_g)}{\partial x} R + 2\sigma C_g \left(\frac{\partial R}{\partial x}\right)\right\} - \frac{D^2 R}{Dt^2} - (\nabla_h \cdot \mathbf{U}) \frac{DR}{Dt} + \nabla_h \cdot (CC_g \nabla_h R) = 0 \tag{26}$$

This equation may be compared to the linearized form of the evolution equation of *Turpin et al.* [1983] by considering one-dimensional propagation with a colinear current, i.e., $\partial/\partial y \equiv 0$, $\mathbf{U} = \{U, 0\}$. Equation (26) then reduces to

$$2i\sigma\left\{\frac{\partial R}{\partial t} + (C_g + U)\frac{\partial R}{\partial x}\right\} + i\frac{\partial \sigma}{\partial t}R + i\frac{\partial}{\partial x}(\sigma(C_g + U))R + \frac{\partial}{\partial x}\left(CC_g\frac{\partial R}{\partial x}\right) - \frac{\partial^2 R}{\partial t^2} - \frac{\partial}{\partial t}\left(U\frac{\partial R}{\partial x}\right) - \frac{\partial}{\partial x}\left(U\frac{\partial R}{\partial t}\right) - \frac{\partial}{\partial x}\left(U^2\frac{\partial R}{\partial x}\right) = 0$$

The linearized form of the evolution equation governing the wave amplitude $A$ follows by substituting for $R$:

$$i\left\{\frac{\partial A}{\partial t} + (C_g + U)\frac{\partial A}{\partial x}\right\} + \frac{i}{2}\left\{\sigma\frac{\partial}{\partial x}\left(\frac{C_g + U}{\sigma}\right) - \frac{1}{\sigma}\frac{\partial \sigma}{\partial t}\right\}A + \frac{1}{2}\frac{\partial}{\partial x}\left\{CC_g\frac{\partial}{\partial x}\left(\frac{A}{\sigma}\right)\right\} - \frac{1}{2}\frac{D^2}{Dt^2}\left(\frac{A}{\sigma}\right) - \frac{1}{2}\frac{\partial U}{\partial x}\frac{D}{Dt}\left(\frac{A}{\sigma}\right) = 0 \quad (27)$$

The first two terms of (27) correctly model the transport of wave action with the absolute group velocity $(C_g + U)$ in the variable domain, and are equivalent to the leading order result of Turpin et al. in a stationary reference frame, which was obtained directly from the equations of motion using a multiple scale expansion. The remaining terms represent dispersive effects. Equation (26) represents the linearized, slow modulation counterpart of the formulation of *Djordjevic and Redekopp* [1978] extended to two horizontal dimensions and a moving domain.

The parabolic approximation for linear plane waves in two dimensions may be obtained from (26) by neglecting time dependence and assuming that $\partial/\partial x \sim \delta(\partial/\partial y)$. After further neglecting terms containing $U^2$ in comparison to $CC_g$ (applicable far from stopping points), the parabolic equation is given by

$$2ikA_x + 2ik\left(\frac{V}{C_g + U}\right)A_y + i\frac{k\sigma}{(C_g + U)} \cdot \left\{\frac{\partial}{\partial x}\left(\frac{C_g + U}{\sigma}\right) + \frac{\partial}{\partial y}\left(\frac{V}{\sigma}\right)\right\}A + \frac{k}{\sigma(C_g + U)}(CC_gA_y)_y = 0 \quad (28)$$

Results obtained using (28) and its weakly nonlinear counterpart have been discussed by *Kirby and Dalrymple* [1983*a*], while the case of weakly nonlinear motion in the absence of currents has been investigated by *Kirby and Dalrymple* [1983*b*].

## 5. CONCLUSIONS

A wave equation governing the propagation of linearized waves in regions with varying depth and current has been derived. This equation differs slightly from the equation given by *Booij* [1981] due to errors in his derivation, and also differs in a crucial manner from the equation of *Liu* [1983]. The validity of the present equation has been demonstrated, first, by showing that it correctly leads to the conservation equation for wave action without further approximation, and second, by demonstrating equivalence to the linearized form of the evolution equation of *Turpin et al.* [1983], which is obtained directly by perturbation expansion of the governing equations. Additionally, a parabolic equation approximation for linear waves on a current has also been provided.

*Acknowledgments.* This research was supported in part by the Office of Naval Research, Coastal Sciences Program. Discussions with R. A. Dalrymple are gratefully acknowledged.

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(Received May 25, 1983;
revised June 30, 1983;
accepted July 1, 1983.)