Oblique envelope solutions of the Davey–Stewartson equations in intermediate water depth

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Various types of envelope solutions of modulated Stokes waves may exist depending on the relative signs of terms representing dispersive and nonlinear effects. In this study, the variation of the zeroes of the dispersive and nonlinear terms in the equation governing evolution of the envelope is determined as a function of dimensionless water depth kh and oblique angle α, and solitary and periodic oblique envelope solutions are described for the general case of intermediate water depth.

I. INTRODUCTION

The propagation and evolution of amplitude modulations of weakly nonlinear Stokes waves in time and physical space $\{x,y\}$ can be described by the coupled nonlinear Schrödinger–Poisson equations given by Davey and Stewartson,[1] which, in a stationary reference frame and for constant water depth, may be written as

$$i(A_t + C_g A_x) - \frac{C_g^2}{2\omega} G A_{xx} + \frac{CC_g}{2\omega} A_{yy} - \frac{1}{2}\omega k^2 D |A|^2 A + \left(\frac{k^2}{2\omega \cosh^2 kh}\tilde{\phi}_t - k\tilde{\phi}_x\right) A = 0, \tag{1}$$

and

$$\tilde{\phi}_{tt} - gh(\tilde{\phi}_{xx} + \tilde{\phi}_{yy}) = \frac{\omega^2}{2}\left(\frac{\omega}{k \tanh^2 kh} + \frac{C_g}{2\sinh^2 kh}\right)|A|^2_x. \tag{2}$$

Here, A represents the $O(\epsilon)$ complex amplitude of the leading-order wave, where ϵ is the Stokes steepness parameter kA. The water surface is described by

$$\eta = \mathrm{Re}\{A \exp[i(kx - \omega t)]\},$$

and $\tilde{\phi}$ is the potential for the $O(\epsilon^2)$ wave-induced motion. The coefficients are determined according to

$$\omega^2 = gk \tanh kh, \tag{3}$$

where ω is the wave angular frequency; further,

$$D = (\cosh 4kh + 8 - 2\tanh^2 kh)/8\sinh^4 kh,$$
$$G = 1 - 2\omega^2 h(1 - kh \tanh kh)/C_g^2 k \sinh 2kh, \tag{4}$$
$$D, G \to 1 \quad \text{as} \quad kh \to \infty,$$

and C and C_g are the linear phase and group velocities given by $C = \omega/k$ and $C_g = \partial\omega/\partial k$. In the limit of large kh, (1) and (2) decouple to give

$$i\left(A_t + \frac{\omega}{2k}A_x\right) - \frac{\omega}{8k^2}A_{xx} + \frac{\omega}{4k^2}A_{yy} - \frac{\omega k^2}{2}|A|^2 A = 0, \tag{5}$$

and an unforced wave equation for $\tilde{\phi}$.[2,3]

For the purpose of a study on shoaling of modulated nonlinear wave trains, it was desirable to extend the known deepwater results for solutions of Eqs. (1) and (2) to the case of intermediate and locally constant depth. We study the special case of one-dimensional envelopes, where the fundamental wave propagates in the x direction and the envelope propagates in direction z given by

$$z = x\cos\alpha + y\sin\alpha, \tag{6}$$

where $\alpha = 0$ corresponds to the one-dimensional case. Solutions for $\alpha = 0$ and arbitrary kh have been given by Hasimoto and Ono,[2] while Saffman and Yuen[3] and Hui and Hamilton[4] have studied the case of arbitrary α in the limit of $kh \to \infty$ [deepwater, Eq. (5)].

II. ENVELOPE EQUATIONS AND ZEROES OF THE COEFFICIENTS

We proceed by transforming Eqs. (1) and (2) to the z coordinate, yielding the coupled set

$$i(A_t + C_g\cos\alpha A_z) - \mu A_{zz} - \tfrac{1}{2}\omega k^2 D|A|^2 A + (k^2\tilde{\phi}_t/2\omega\cosh^2 kh - k\cos\alpha\,\tilde{\phi}_z)A = 0, \tag{7}$$

$$\tilde{\phi}_{tt} - gh\tilde{\phi}_{zz} = \frac{\omega^2}{2}\cos\alpha\left(\frac{\omega}{k\tanh^2 kh} + \frac{C_g}{2\sinh^2 kh}\right)|A|^2_z, \tag{8}$$

where

$$\mu = (C_g^2/2\omega)G\cos^2\alpha - (CC_g/2\omega)\sin^2\alpha. \tag{9}$$

Adopting a coordinate system moving with the envelope given by

$$\tau = \epsilon t,$$
$$\xi = z - C_g\cos\alpha t, \tag{10}$$

transforms (7) and (8) to the form

$$iA_\tau - \mu A_{\xi\xi} - \tfrac{1}{2}\omega k^2 D|A|^2 A + \left[\frac{k^2}{2\omega\cosh^2 kh}\tilde{\phi}_\tau - k\cos\alpha \times\left(\frac{kC_g}{2\omega\cosh^2 kh} + 1\right)\tilde{\phi}_\xi\right]A = 0, \tag{11}$$

$$\tilde{\phi}_{\tau\tau} - C_g \cos\alpha \tilde{\phi}_{\tau\xi} + (C_g^2 \cos^2\alpha - gh)\tilde{\phi}_{\xi\xi} = \frac{g^2 k \cos\alpha}{2\omega}\left(\frac{kC_g}{2\omega\cosh^2 kh} + 1\right)|A|^2_{\xi}. \qquad (12)$$

In the one-dimensional case being studied, $\tilde{\phi}$ may be eliminated from Eq. (11) by assuming that $\tilde{\phi}$ is entirely forced by the fundamental wave train and is thus steady in the moving coordinate system. Dropping derivatives with respect to τ in Eq. (12) and integrating once over ξ yields the result

$$\tilde{\phi}_{\xi} = \frac{g^2 k\cos\alpha}{2\omega(C_g^2\cos^2\alpha - gh)} \times\left(\frac{kC_g}{2\omega\cosh^2 kh} + 1\right)|A|^2 + Q_0(\tau), \qquad (13)$$

where $Q_0(\tau)$ is an integration constant. For wave fields where $A \to 0$ as $\xi \to \pm\infty$ in a domain otherwise at rest, Q_0 may be uniquely determined as being equal to zero. For wave fields with finite amplitude A as $\xi \to \pm\infty$, Q_0 is formally arbitrary without the imposition of boundary conditions, but is presumed to be small enough not to enter the dispersion relation (3) directly. We proceed by assuming Q_0 to be steady in the moving frame. Substituting Eq. (13) into Eq. (11) then gives the single equation

$$iA_\tau = \mu A_{\xi\xi} - \nu|A|^2 A - QA = 0, \qquad (14)$$

where

$$\nu = \frac{\omega k^2}{2}\left[D + \frac{\omega^2\cos^2\alpha}{k^2\tanh^2 kh\,(C_g^2\cos^2\alpha - gh)} \times\left(\frac{kC_g}{2\omega\cosh^2 kh} + 1\right)^2\right]. \qquad (15)$$

The coefficients μ and ν have the well-known deepwater limits[3]

$$\mu = (\omega/8k^2)(1 - 3\sin^2\alpha),$$

$$\nu = \omega k^2/2.$$

The effect of Q may be incorporated in the envelope A by the substitution

$$A = Be^{-iQ\tau},$$

yielding the modified governing equation

$$iB_\tau - \mu B_{\xi\xi} - \nu|B|^2 B = 0. \qquad (16)$$

The types of solutions of permanent form applicable to Eq. (16) are determined by the sign of the product $\mu\nu$, and thus the boundaries of regions in the kh-α plane in which any solution type may exist are determined by the zeroes of μ and ν. Zeroes of μ, given by Eq. (9), are determined by the relation

$$\alpha(\mu = 0) = \tan^{-1}[(C_g/C)G]^{1/2}, \qquad (17)$$

with $\mu > 0$ for $\alpha < \alpha(\mu = 0)$. The familiar deepwater limit is given by[3] $\alpha(\mu = 0) = 35.26°$. From Eq. (15) the condition that $\nu = 0$ is given by

$$\alpha(\nu = 0) = \cos^{-1}\left(\frac{kh}{\tanh kh}\frac{D/H}{[1 + (C_g/C)^2(D/H)]}\right)^{1/2}, \qquad (18)$$

where

$$H = \frac{1}{\tanh^2 kh}\left(\frac{kC_g}{2\omega\cosh^2 kh} + 1\right)^2.$$

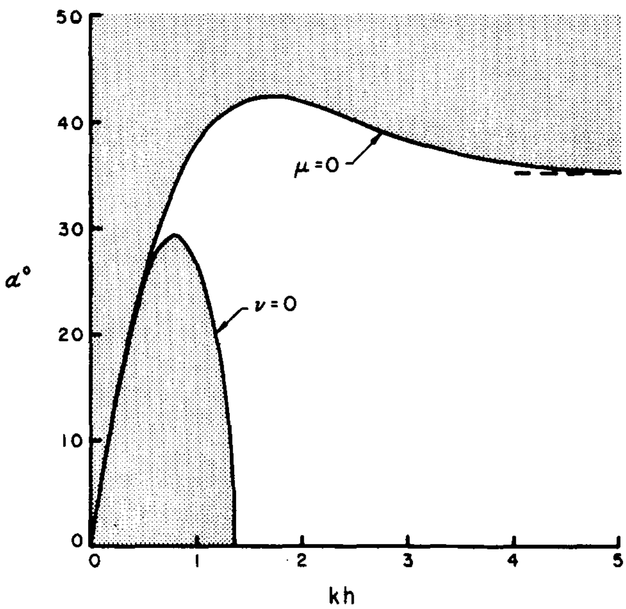

FIG. 1. The kh-α plane. Oblique plane solitons are excluded from shaded regions. The dashed line is the deepwater asymptote for $\mu = 0$, given by $\alpha = 35.26°$.

The parameter ν obtains positive values for $\alpha > \alpha(\nu = 0)$. Equation (18) has no roots for $kh > 1.36$, and $\nu > 0$ for all α in this range. The curves given by Eqs. (17) and (18) are plotted in Fig. 1. The curves intersect at $kh = 0.37$, $\alpha = 19.61°$, and a narrow region with $\mu\nu > 0$ exists between the curves for $kh < 0.37$. Equation (18) has a maximum at $kh = 0.78$, corresponding to $\alpha = 29.37°$. Above the curve for $\nu = 0$, the forced wave $\tilde{\phi}$ opposing the propagating group is never strong enough to overcome the amplitude dispersion induced by the cubic nonlinearity. However, the range where this effect may allow the case 1 solutions described below to exist in relatively shallow water is severely limited by the condition (17) on μ. In Fig. 1, shaded regions indicate values of $\mu\nu < 0$, while unshaded regions correspond to $\mu\nu > 0$.

III. ENVELOPE SOLUTIONS OF STEADY FORM

Solutions of Eq. (16) in terms of Jacobian elliptic functions may be obtained following the procedure of Hui and Hamilton.[4] The amplitude B is written in terms of real-valued functions of ξ and τ according to

$$B = \tilde{B}(\xi)e^{-i\sigma\tau}. \qquad (19)$$

Substituting Eq. (19) into Eq. (16) and defining E to be the energy-like quantity

$$E = \tilde{B}^2,$$

leads to the standard form

$$(\gamma/8)(E_\xi)^2 = -E^3 + \beta E^2 + (C/4)E, \qquad (20)$$

where

$$\beta = 2\sigma/\nu, \quad \gamma = 4\mu/\nu,$$

and where C is an arbitrary constant of integration. We consider the two cases of $\gamma \gtrless 0$.

Case 1: $\gamma > 0$ ($\mu\nu > 0$; unshaded regions in Fig. 1)

The following two solutions exist in terms of elliptic functions:

(a) $C = b^2 > 0$. Equation (20) may be written as

$$(\gamma/8)(E_\xi)^2 = E(E_2 - E)(E - E_1), \quad E_2 > E > 0, \tag{21}$$

where

$$E_2 = \tfrac{1}{2}[\beta + (\beta^2 + b^2)^{1/2}] > 0,$$
$$E_1 = \tfrac{1}{2}[\beta - (\beta^2 + b^2)^{1/2}] < 0.$$

The solution of Eq. (21) is given by

$$E = E_2 \operatorname{cn}^2[(2/\gamma)^{1/2}(\beta^2 + b^2)^{1/4}\xi],$$

with modulus m given by

$$m = [E_2/(E_2 - E_1)]^{1/2}.$$

The corresponding solution for $B(\xi,\tau)$ is given by

$$B = E_2^{1/2} \operatorname{cn}[(\nu/2\mu)^{1/2}(E_2 - E_1)^{1/2}\xi] e^{-i(\nu/2)(E_2 + E_1)\tau}. \tag{22}$$

(b) $-\beta^2 < -b^2 = C < 0$. Factoring Eq. (20) results again in Eq. (21), with

$$E_2 \geqslant E \geqslant E_1,$$

where

$$E_2 = \tfrac{1}{2}[\beta + (\beta^2 - b^2)^{1/2}] > 0,$$
$$E_1 = \tfrac{1}{2}[\beta - (\beta^2 - b^2)^{1/2}] > 0.$$

The solution is given by

$$E = E_1/\{1 - (E_2 - E_1)\operatorname{sn}^2[(2/\gamma)^{1/2}E_2^{1/2}\xi]/E_2\},$$

with modulus given by

$$m = [(E_2 - E_1)/E_2]^{1/2}.$$

The corresponding solution for B is

$$B = E_2^{1/2} \operatorname{dn}[(\nu/2\mu)^{1/2}E_2^{1/2}\xi] e^{-(\nu/2)(E_2 + E_1)\tau}. \tag{23}$$

Both solutions (22) and (23) admit the limit $C \to 0$ ($m \to 1$) yielding the solution of solitary form

$$A(\xi,\tau) = B = A_0 \operatorname{sech}[(\nu/2\mu)^{1/2}A_0\xi] e^{-i(\nu/2)A_0^2\tau}, \tag{24}$$

where $E_2 = A_0^2$ and A_0 is the peak amplitude of the profile. This solution has the deepwater limiting form (in stationary coordinates)

$$A = A_0 \operatorname{sech}\left(\frac{\sqrt{2}k^2A_0}{(1 - 3\sin^2\alpha)^{1/2}}(z - C_g \cos\alpha t)\right) \times \exp\left(-i\frac{(kA_0)^2}{4}\omega t\right) \tag{25}$$

given by Saffman and Yuen.[3] Equation (23) also admits the limit $b^2 \to \beta^2$, $m \to 0$, resulting in plane wave solutions of constant amplitude

$$B = A_0 e^{-i\nu A_0^2\tau}.$$

Case 2: $\gamma < 0$ ($\mu\nu < 0$; shaded regions in Fig. 1)

For this case, either μ or ν may be negative with the other held positive. Equation (20) may be factored as

$$(-\gamma/8)(E_\xi)^2 = E(E_1 - E)(E_2 - E), \quad 0 \leqslant E \leqslant E_1. \tag{26}$$

Bounded solutions of Eq. (26) only exist for $\beta > 0$ and $-\beta^2 < C = -b^2 \leqslant 0$, indicating that σ must change sign based on the value of ν. Thus, for the region above the curve given by Eq. (17), ν is > 0 and $\sigma > 0$. For the region under the curve given by Eq. (18), ν is < 0 and thus $\sigma < 0$ to give $\beta > 0$. This represents the effect of the adverse current $\tilde{\phi}_\xi$, slowing the carrier wave to a speed lower than that predicted by linear theory. Solutions of Eq. (26) are given by

$$E_2 = \tfrac{1}{2}[\beta + (\beta^2 - b^2)^{1/2}],$$
$$E_1 = \tfrac{1}{2}[\beta - (\beta^2 - b^2)^{1/2}],$$
$$E_2, E_1 > 0,$$

and

$$E = E_1 \operatorname{sn}^2[(-2/\gamma)^{1/2}E_2\xi],$$

with modulus m given by

$$m = (E_1/E_2)^{1/2}.$$

The corresponding solution for B is given by

$$B = E_1^{1/2} \operatorname{sn}[(-\nu/2\mu)^{1/2}E_2^{1/2}\xi] e^{-i(\nu/2)(E_2 + E_1)\tau}. \tag{27}$$

Equation (27) admits the limit $b^2 \to \beta^2$, $m \to 1$, yielding the solution

$$B = A_0 \tanh[(-\nu/2\mu)^{1/2}A_0] e^{-i\nu A_0^2\tau}, \tag{28}$$

which represents the propagation of a phase jump.

IV. DISCUSSION

In addition to the solutions discussed above, the two-dimensional equation (16) should admit solutions by the Inverse Scattering Transform in analogy to the two-dimensional problem discussed by Zakharov and Shabat.[5] In particular, the solitary form (24) represents the simplest case of the N-soliton solution.

Recent results [6] indicate that the soliton or soliton-like solutions are unstable to transverse perturbations. Less is known about the stability of the periodic solutions. Recent experimental results[7] indicate the presence of oblique wave groups in a deepwater case, and it is likely that solutions of these forms may play a role at least in the transient evolution of complex wave fields in two space dimensions.

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