

# Simplified higher-order Boussinesq equations I. Linear simplifications

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## Abstract

In this paper, we derive and test simplified higher-order Boussinesq equations, i.e., higher-order Boussinesq equations which only show lower-order terms. Simplifications are performed linearly for flat beds and slopes of  $O(\nabla h)$ . With proper coefficient choice, dispersion and shoaling properties are found to be good, while interior fluid velocities show relatively greater error at high wavenumbers. The resulting sets of equations are found to be variants of already-existing equations, which may be easily modified to improve performance. The new equations have dispersion identical to previous results but significantly improved shoaling. © 2002 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

In the past decade, Boussinesq-type equations for water waves have been the subject of much research. As available computational power steadily increases, they offer the hope of tractable nonlinear nearshore wave models with relatively few approximations. This has led to a flurry of interest in both applications for Boussinesq models and, more to the point of this paper, in extensions and improvements to the theoretical Boussinesq framework.

This rising interest in Boussinesq theory began with improvements in dispersion, and sometimes shoaling, in  $O(\mu^2)$  equations, where  $\mu=(\kappa h)_0$  is a

representative dimensionless wavenumber which is assumed to be small. Madsen et al. (1991) and Madsen and Sørensen (1992) used consistent asymptotic manipulations to obtain a Padé [2,2] approximant to the linear dispersion relationship, along with considerably improved shoaling. In contrast, Nwogu (1993) redefined the representative velocity variable to be the velocity at some arbitrary fraction of the still water depth and also achieved Padé [2,2] dispersion. All of these equations are linear at the highest dispersive order, which has been found to limit their usefulness for highly nonlinear waves. Wei et al. (1995; herein denoted WKGS) extended Nwogu's equations by assuming instead that nonlinearity was strong at dispersive order  $O(\mu^2)$  and found significant increases in accuracy for near-breaking waves. Kennedy et al. (2000a) further improved nonlinear performance in

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the WKGS equations by redefining the concept of reference elevation to include a nonlinear component. Many other authors have written on Boussinesq theory, but the above work, particularly Nwogu's equations and the WKGS equations have direct links to the present work.

All of these sets of equations are  $O(\mu^2)$ , which includes the first set of terms in a Taylor series expansion about the long wave limit. By taking more terms in the Taylor series,  $O(\mu^4)$  equations may be derived, which are presumably more accurate. This approach was used by Madsen and Schäffer (1998) and Gobbi and Kirby (1999; herein denoted GK), among others. In almost every case, it was found that including the extra terms in the Taylor series gave an improved representation of linear and nonlinear analytic and computational properties. However, the resulting equations were also found to be extremely complex, which made their implementation and analysis difficult. Furthermore, the equations contained either fifth or sixth derivatives, which are tricky to compute numerically in the presence of even small noise. These twin concerns have limited the general acceptance of  $O(\mu^4)$  Boussinesq equations.

There have been some direct and indirect attempts to simplify higher order Boussinesq equations. Madsen and Schäffer (1998) found that some  $O(\mu^2)$  Boussinesq equations "enhanced" to improve dispersive and shoaling properties [but still showing only  $O(\mu^2)$  terms] were linearly identical to  $O(\mu^4)$  Boussinesq equations for flat beds and small slopes in one dimension. Two-dimensional effects were not consid-

ered. Zou (2000) enhanced some different higher-order Boussinesq equations and cancelled linear terms, leading to equations which were of  $O(\mu^4)$  on a flat bed but again only showed  $O(\mu^2)$  terms. These equations were linearly identical to the Madsen and Schäffer (1998, Section 6) enhanced equations on a flat bed, which had been derived as  $O(\mu^2)$  equations.

We will continue in this direction by deriving and analysing several sets of simplified higher-order Boussinesq-type equations for water waves. The new sets of equations will be shown to bear strong resemblances to  $O(\mu^2)$  equations currently in wide use, namely Nwogu's (1993) equations (or WKGS, which are linearly identical) and the "enhanced" equations of Madsen and Schäffer (1998, Section 6; herein denoted MS), which are based on the WKGS equations. Only linear simplifications will be considered and, thus, we will not consider any nonlinear properties except in an indirect sense, from knowledge of previous studies. In their different ranges of validity, the two new sets of equations will show very good properties for dispersion and shoaling, while fluid velocities in the water column show somewhat more error.

When compared with the  $O(\mu^2)$  equations they closely resemble, the new equations will be identical on a flat bottom, but not on a sloping bed. Thus, linear dispersion will be identical, but shoaling properties will be different, and much improved.

Finally, we will show that in some cases  $O(\mu^4)$  equations are actually  $O(\mu^8)$ , and in one special case, a seemingly  $O(\mu^2)$  equation is  $O(\mu^8)$  on a flat bed.

## 2. Two new sets of Boussinesq-type equations

In this section, we create two new sets of simplified higher-order Boussinesq equations from the  $O(\mu^4)$  equations of Gobbi and Kirby (1999) and Gobbi et al. (2000a). All simplifications will be done linearly.

### 2.1. Governing equations and basic scaling

First, we will begin with some scaling. Asterisks here denote dimensionless variables.

$$(x^*, y^*) = \kappa_0(x, y), \quad z^* = \frac{z}{h}, \quad t^* = \kappa_0 \sqrt{gh_0} t, \quad h^* = \frac{h}{h_0}, \quad \eta^* = \frac{\eta}{\delta h_0},$$

$$(u^*, v^*) = \frac{1}{\delta \sqrt{gh_0}}(u, v), \quad w^* = \frac{\mu}{\delta \sqrt{gh_0}} w \quad (1)$$

where  $(x, y, z)$  are the horizontal and vertical coordinates,  $t$  is time,  $h$  is the still water depth,  $\eta$  is the free surface elevation and  $(u, v, w)$  are fluid velocities in the three spatial coordinates. Here,  $\mu = \kappa_0 h_0$  is a typical dimensionless wavenumber which is assumed to be small and  $\delta = a_0/h_0$  is a measure of nonlinearity.

Dropping the asterisks on the dimensionless variables from now on, and defining  $\mathbf{u} \equiv (u, v)$ , the fluid flow is subject to the continuity equation:

$$\mu^2 \nabla \cdot \mathbf{u} + w_z = 0 \quad (2)$$

throughout the entire flow field and a tangential flow condition:

$$\mu^2 \nabla h \cdot \mathbf{u} + w = 0, \quad z = -h(x, y) \quad (3)$$

on the bottom boundary. As we are dealing primarily with wave motion, we will also impose the condition of no horizontal vorticity throughout the field,

$$u_z - w_x = 0, \quad w_y - v_z = 0 \quad (4)$$

Fluid motion evolves in time subject to kinematic and dynamic boundary conditions at the free surface:

$$\eta_t + \mathbf{u} \cdot \nabla \eta = w, \quad z = \delta \eta \quad (5)$$

$$\frac{D\mathbf{u}}{Dt} + \nabla \eta \left( 1 + \frac{Dw}{Dt} \right) = 0, \quad z = \delta \eta \quad (6)$$

where Eq. (6) is only valid at the free surface,  $\nabla = (\partial/\partial x, \partial/\partial y)$  and  $D/Dt = (\partial/\partial t) + \mathbf{u} \cdot \nabla + (1/\mu^2)w(\partial/\partial z)$ . The kinematic free surface boundary condition (5) is usually transformed, through continuity and the bottom boundary condition, into an explicit statement of mass conservation:

$$\eta_t + \nabla \cdot M = 0 \quad (7)$$

where

$$M = \int_{-h}^{\delta \eta} \mathbf{u} dz \quad (8)$$

## 2.2. The GK equations

The linearised GK equations, keeping all slope terms up to  $O(\mu^2)$  and  $O(\mu^4)$  terms up to slopes of  $O(\nabla h)$  are:

$$\begin{aligned} \eta_t + \nabla \cdot (h\tilde{\mathbf{u}}) + \mu^2 \nabla \cdot \left\{ h^2 \left( A - \frac{1}{2} \right) (\nabla(\nabla \cdot (h\tilde{\mathbf{u}})) - h\nabla(\nabla \cdot \tilde{\mathbf{u}})) + \frac{1}{2} h^3 \left( B - \frac{1}{3} \right) \nabla(\nabla \cdot \tilde{\mathbf{u}}) \right\} \\ + \frac{\mu^4}{4} h^5 \left( B^2 - \frac{B}{3} - \frac{D}{6} + \frac{1}{30} \right) \nabla^2 \nabla^2 \nabla \cdot \tilde{\mathbf{u}} + \mu^4 \left[ B \left( A - \frac{1}{2} \right) + \left( B - \frac{1}{3} \right) \left( A + \frac{9}{4} B \right) \right. \\ \left. - \frac{2}{3} \left( C - \frac{1}{4} \right) - \frac{5}{24} \left( D - \frac{1}{5} \right) \right] Q_1 = 0 \end{aligned} \quad (9)$$

$$\begin{aligned}
\tilde{\mathbf{u}}_t + \nabla\eta + \mu^2[(A-1)h(\nabla(\nabla \cdot (h\tilde{\mathbf{u}})) - h\nabla(\nabla \cdot \tilde{\mathbf{u}}))]_t + \frac{\mu^2}{2}[h^2(B-1)\nabla(\nabla \cdot \tilde{\mathbf{u}})]_t \\
+ \frac{\mu^4}{4}h^4\left(B^2 - B - \frac{D}{6} + \frac{1}{6}\right)\nabla\nabla^2(\nabla \cdot \tilde{\mathbf{u}}_t) + \mu^4\left[(A-1)\frac{B}{2} + (B-1)\frac{B}{2} - \frac{1}{6}(C-1)\right]\mathbf{R}_1 \\
+ \left[(A-1)\frac{B}{2} + (B-1)\left(A + \frac{B}{2}\right) - \frac{1}{2}(C-1)\right]\mathbf{S}_1 = 0
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
Q_1 &\equiv h^4\nabla h \cdot \nabla\nabla^2\nabla \cdot \tilde{\mathbf{u}} \\
\mathbf{R}_1 &\equiv h^3\nabla h\nabla^2\nabla \cdot \tilde{\mathbf{u}}_t \\
\mathbf{S}_1 &\equiv h^3\nabla(\nabla h \cdot \nabla(\nabla \cdot \tilde{\mathbf{u}}_t))
\end{aligned} \tag{11}$$

and we only keep terms up to  $O(\nabla h)$  in  $\mathbf{S}_1$ . It was necessary to assume irrotationality in the horizontal plane at  $O(\mu^4\nabla h)$  to write the equations in this form.

The GK equations have four free coefficients  $A$ – $D$ , which GK used to obtain a dispersion relation that was a Padé [4,4] approximant to the full linear solution. We redefine the coefficients to achieve the greater generality necessary for our work and arrive at:

$$\tilde{\mathbf{u}} \equiv \frac{1}{4}[\mathbf{u}(z = z_{\alpha 1}) + \mathbf{u}(z = z_{\alpha 2}) + \mathbf{u}(z = z_{\alpha 3}) + \mathbf{u}(z = z_{\alpha 4})] \tag{12}$$

$$Ah \equiv \frac{1}{4}[(h + z_{\alpha 1}) + (h + z_{\alpha 2}) + (h + z_{\alpha 3}) + (h + z_{\alpha 4})] \tag{13}$$

$$Bh^2 \equiv \frac{1}{4}[(h + z_{\alpha 1})^2 + (h + z_{\alpha 2})^2 + (h + z_{\alpha 3})^2 + (h + z_{\alpha 4})^2] \tag{14}$$

$$Ch^3 \equiv \frac{1}{4}[(h + z_{\alpha 1})^3 + (h + z_{\alpha 2})^3 + (h + z_{\alpha 3})^3 + (h + z_{\alpha 4})^3] \tag{15}$$

$$Dh^4 \equiv \frac{1}{4}[(h + z_{\alpha 1})^4 + (h + z_{\alpha 2})^4 + (h + z_{\alpha 3})^4 + (h + z_{\alpha 4})^4] \tag{16}$$

where  $z_{\alpha 1}$ , etc., are four different reference elevations similar to those defined by Nwogu (1993). This does not affect the form of the resulting equations, but now gives us four independent coefficients, while the GK definitions (Eqs. (18)–(21) in GK) only yield three degrees of freedom. Adding more reference levels would not yield any additional degrees of freedom at  $O(\mu^4)$  although, as will be seen, it will at higher levels of approximation.

Our goal is to eliminate all  $O(\mu^4)$  terms in Eqs. (9) and (10) while retaining  $O(\mu^4)$  accuracy. Simplification will be performed using two different techniques: coefficient choice and “enhancement.” Each will allow us to arrive at new sets of Boussinesq-type equations which are formally  $O(\mu^4)$ , but only show  $O(\mu^2)$  terms.

### 2.3. Simplification through coefficient choice

Conceptually, it is easiest to arrive at simplified higher-order equations by choosing coefficients such that higher-order terms disappear. We have four free coefficients  $A$ – $D$  which we may choose freely in order to eliminate  $O(\mu^4)$  terms in Eqs. (9) and (10). Such an approach was pursued by Schäffer and Madsen (1995a), who only considered one horizontal dimension with equations that were, in a different form, linearly equivalent to the GK equations.

On a flat bed, mass and momentum Eqs. (9) and (10) give constraints:

$$\left(B^2 - \frac{B}{3} - \frac{D}{6} + \frac{1}{30}\right) = 0 \quad (17)$$

$$\left(B^2 - B - \frac{D}{6} + \frac{1}{6}\right) = 0 \quad (18)$$

These yield the set ( $B = 1/5$ ,  $D = 1/25$ ), which will provide  $O(\mu^4)$  behaviour on a flat bed while only showing  $O(\mu^2)$  terms.

However, on a sloping bed of  $O(\nabla h)$ , the situation is more complicated. The additional conditions become:

$$\left[B\left(A - \frac{1}{2}\right) + \left(B - \frac{1}{3}\right)\left(A + \frac{9}{4}B\right) - \frac{2}{3}\left(C - \frac{1}{4}\right) - \frac{5}{24}\left(D - \frac{1}{5}\right)\right] = 0 \quad (19)$$

$$\left[(A - 1)\frac{B}{2} + (B - 1)\frac{B}{2} - \frac{1}{6}(C - 1)\right] = 0 \quad (20)$$

$$\left[(A - 1)\frac{B}{2} + (B - 1)\left(A + \frac{B}{2}\right) - \frac{1}{2}(C - 1)\right] = 0 \quad (21)$$

for a total of three constraints. However, only two degrees of freedom ( $A$ ,  $C$ ) are left in order to fulfill these three constraints. Since these three equations are linearly independent, it becomes necessary to make one further assumption: we will assume that higher-order terms are cancelled only for one-dimensional motion. This relaxation means that terms  $\mathbf{R}_1$  and  $\mathbf{S}_1$  are identical and, thus, Eqs. (20) and (21) may be added to give one equation. The two remaining equations may be solved to give the full set of coefficients:

$$(A = 2/5, B = 1/5, C = 1/10, D = 1/25) \quad (22)$$

Using different equations, and considering only one-dimensional motion, a similar result has been given by Schäffer and Madsen (1995a).

Our linearised, simplified  $O(\mu^4)$  equations will, thus, be Eqs. (9) and (10) without  $O(\mu^4)$  terms. These equations will provide linearly  $O(\mu^4)$  behaviour on a flat bed, and on a sloping bed of  $O(\nabla h)$  only for motion perpendicular to bathymetric contours. It, thus, becomes impossible using coefficient choice to give a set of equations only showing  $O(\mu^2)$  terms but of  $O(\mu^4)$  accuracy for flat beds and general slopes of  $O(\nabla h)$ . This is somewhat disappointing, but later shoaling analysis will demonstrate that these equations still provide higher-order asymptotic shoaling behaviour even when waves are not perpendicular to bathymetric contours.

On a flat bed, these equations are identical to the linearised equations of Nwogu (1993) using  $\alpha \equiv (z_\alpha/h)^2/2 + z_\alpha/h = -2/5$ , showing that Nwogu's equations are actually higher order for this particular coefficient choice. This observation has also been made by Madsen and Schäffer (1998).

There is one more item of concern—it turns out that the reference elevations  $z_{\alpha 1} - z_{\alpha 4}$  (Eq. (16)) arising from the set of coefficients ( $A = 2/5$ ,  $B = 1/5$ ,  $C = 1/10$ ,  $D = 1/25$ ) are complex. This does not affect the formal validity of the equations, as all velocities are real, but raises some doubts about their physical interpretation.

#### 2.4. Simplification through enhancement

It is well known that within the order of approximation, dispersive terms in both the mass and momentum equations may be replaced and rearranged to improve performance. As shown by Madsen and Schäffer (1998), among others, this can be equivalent to applying differential operators with free coefficients to the mass and momentum equations. They applied the differential operators:

$$1 + (\psi_2 - \psi_1)\mu^2\nabla \cdot (h^2\nabla) + \psi_2\mu^2\nabla^2(h^2)$$

$$1 + (\rho_2 - \rho_1)\mu^2h^2\nabla(\nabla \cdot) - \rho_2\mu^2h\nabla(\nabla \cdot h) \quad (23)$$

to the WKS equations, which are linearly identical to Nwogu's equations. With the preferred coefficient choice, the resulting system had excellent Padé [4,4] dispersion properties with good nonlinear characteristics on a flat bed. Shoaling performance was somewhat less accurate, as will be shown in Section 4. The overall system was assumed to be  $O(\mu^2)$ .

We will modify their approach and instead work backward: we will start with the  $O(\mu^4)$  GK equations, and cancel higher-order linear terms using a somewhat different operator than that used by MS. The resulting equations will be formally of  $O(\mu^4)$  on flat beds and for slopes of  $O(\nabla h)$  but will only show  $O(\mu^2)$  terms. At  $O(\mu^4\nabla h)$ , we are still restricted to irrotational motion in the horizontal plane.

The differential operator we use will be somewhat different than that of MS. We choose the operator:

$$1 + \psi_1\mu^2\nabla \cdot (h^2\nabla) + \psi_2\mu^2\nabla \cdot (h\nabla h) \quad (24)$$

to apply to the conservation of mass equation, and:

$$1 + \mu^2\rho_1h^2\nabla(\nabla \cdot) + \mu^2\rho_2h\nabla h(\nabla \cdot) + \mu^2\rho_3\nabla(h\nabla h) \quad (25)$$

to the momentum equation, where the  $(\psi_n, \rho_n)$  terms are initially free coefficients.

After some algebra, and keeping terms up to  $O(\mu^4\nabla h)$ , the resulting mass and momentum equations are:

$$\eta_t + \text{unchanged terms up to } O(\mu^2) + \mu^4 \left\{ \frac{1}{4} \left( B^2 - \frac{B}{3} - \frac{D}{6} + \frac{1}{30} \right) + \psi_1 \frac{1}{2} \left( B - \frac{1}{3} \right) \right\} h^5 \nabla^2 \nabla^2 \nabla \cdot \tilde{\mathbf{u}}$$

$$+ \mu^4 \left\{ \left[ B \left( A - \frac{1}{2} \right) + \left( B - \frac{1}{3} \right) \left( A + \frac{9}{4} B \right) - \frac{2}{3} \left( C - \frac{1}{4} \right) - \frac{5}{24} \left( D - \frac{1}{5} \right) \right] \right.$$

$$\left. + \psi_1 \left[ 2 \left( A - \frac{1}{2} \right) + \frac{11}{2} \left( B - \frac{1}{3} \right) \right] + \psi_2 \frac{1}{2} \left( B - \frac{1}{3} \right) \right\} Q_1 = 0 \quad (26)$$

$$\tilde{\mathbf{u}}_t + \text{unchanged terms up to } O(\mu^2) + \mu^4 \left\{ \frac{1}{4} \left( B^2 - B - \frac{D}{6} + \frac{1}{6} \right) + \rho_1 \frac{1}{2} (B - 1) \right\} h^4 \nabla \nabla^2 (\nabla \cdot \tilde{\mathbf{u}}_t)$$

$$+ \mu^4 \left\{ \left[ (A - 1) \frac{B}{2} + (B - 1) \frac{B}{2} - \frac{1}{6} (C - 1) \right] + \rho_1 (B - 1) + \rho_2 \frac{1}{2} (B - 1) \right\} \mathbf{R}_1$$

$$+ \mu^4 \left\{ \left[ (A - 1) \frac{B}{2} + (B - 1) \left( A + \frac{B}{2} \right) - \frac{1}{2} (C - 1) \right] + \rho_1 [2(A - 1) + (B - 1)] + \rho_3 \frac{1}{2} (B - 1) \right\} \mathbf{S}_1 = 0 \quad (27)$$

To cancel all higher-order terms, we require:

$$\psi_1 = \frac{-\left[\left(B - \frac{1}{3}\right)\frac{B}{4} - \frac{1}{24}\left(D - \frac{1}{5}\right)\right]}{\frac{1}{2}\left(B - \frac{1}{3}\right)}$$

$$\rho_1 = \frac{-\left[(B - 1)\frac{B}{4} - \frac{1}{24}(D - 1)\right]}{\frac{1}{2}(B - 1)}$$

$$\psi_2 = \frac{-\left[B\left(A - \frac{1}{2}\right) + \left(B - \frac{1}{3}\right)\left(A + \frac{9}{4}B\right) - \frac{2}{3}\left(C - \frac{1}{4}\right) - \frac{5}{24}\left(D - \frac{1}{5}\right)\right] - \psi_1\left[2\left(A - \frac{1}{2}\right) + \frac{11}{2}\left(B - \frac{1}{3}\right)\right]}{\frac{1}{2}\left(B - \frac{1}{3}\right)}$$

$$\rho_2 = \frac{-\rho_1(B - 1) - \left[(A - 1)\frac{B}{2} + (B - 1)\frac{B}{2} - \frac{1}{6}(C - 1)\right]}{\frac{1}{2}(B - 1)}$$

$$\rho_3 = \frac{-\rho_1[2(A - 1) + (B - 1)] - \left[(A - 1)\frac{B}{2} + (B - 1)\left(A + \frac{B}{2}\right) - \frac{1}{2}(C - 1)\right]}{\frac{1}{2}(B - 1)} \quad (28)$$

Using these definitions, the coefficients  $A$ – $D$  remain free, as all  $O(\mu^4)$  terms are cancelled by specifying  $\psi_1$ , etc. This allows us to choose the remaining coefficients in order to best improve linear dispersion and shoaling.

If we were only concerned with the equations on a flat bed, we could have used the MS operators (Eq. (23)) directly. However, on a varying bed, the necessity of cancelling in two dimensions both  $\mathbf{R}_1$  and  $\mathbf{S}_1$  terms means that the slightly more complex (Eq. (25)) was necessary. The MS operators could have been used unchanged to cancel terms in one horizontal dimension only.

The form of the operators used here is not unique—there appear to be numerous operators that could have been used to cancel higher-order terms. The specific operators used here (Eqs. (24) and (25)) were chosen as they simplified the shoaling analysis considerably. It is also possible that another set of operators could also remove the irrotationality assumption which was needed at  $O(\mu^4\nabla h)$ . However, none was immediately forthcoming.

The resulting equations (hereafter denoted Set 2) have  $O(\mu^4)$  validity in two dimensions on a flat bed and on slopes of  $O(\nabla h)$ . They resemble closely the MS equations—on a flat bed they are identical for an appropriate choice of coefficients. They are also linearly equivalent to the equations of Zou (2000) on a flat bed. However, on slopes of  $O(\nabla h)$ , they differ from both of these sets—the MS equations have a different operator as noted earlier, while the Zou equations are formally only  $O(\mu^2)$  accurate in this case. The present equations (Set 2) will be shown to have improved performance on sloping beds due to their full cancellation of higher-order slope terms of  $O(\nabla h)$ .

Four coefficients  $A$ – $D$  remain to be specified. Two ( $B, D$ ) will be used to optimise linear dispersion, while those remaining ( $A, C$ ) will be used to improve shoaling.

### 3. Linear properties

Linearly, both sets 1 (coefficient choice) and 2 (enhancement) may be represented by the same mass and momentum equations with different coefficients. For set 1, coefficients  $A$ – $D$  are given by (Eq. (22)), while enhancement coefficients  $\psi_1$ , etc., are zero. For set 2, enhancement coefficients are given by (Eq. (28)), while  $A$ – $D$  are still free.

The linearised mass and momentum equations have the form:

$$\eta_t + \nabla \cdot (h\tilde{\mathbf{u}}) + \mu^2 \nabla \cdot \left\{ h^2 \left( A - \frac{1}{2} \right) (\nabla(\nabla \cdot (h\tilde{\mathbf{u}})) - h\nabla(\nabla \cdot \tilde{\mathbf{u}})) + \frac{1}{2} h^3 \left( B - \frac{1}{3} \right) \nabla(\nabla \cdot \tilde{\mathbf{u}}) + \psi_1 h^2 \nabla N_1 + \psi_2 h \nabla h N_1 \right\} = 0 \quad (29)$$

$$\tilde{\mathbf{u}}_t + \nabla \eta + \mu^2 \{ (A - 1) h (\nabla(\nabla \cdot (h\tilde{\mathbf{u}}_t)) - h\nabla(\nabla \cdot \tilde{\mathbf{u}}_t)) + \frac{1}{2} h^2 (B - 1) \nabla(\nabla \cdot \tilde{\mathbf{u}}_t) + \rho_1 h^2 \nabla(\nabla \cdot \mathbf{O}_1) + \rho_2 h \nabla h (\nabla \cdot \mathbf{O}_1) + \rho_3 \nabla(h \nabla h \cdot \mathbf{O}_1) \} = 0 \quad (30)$$

where

$$N_1 \equiv \eta_t + \nabla \cdot (h\tilde{\mathbf{u}}) \quad (31)$$

$$\mathbf{O}_1 \equiv \tilde{\mathbf{u}}_t + \nabla \eta \quad (32)$$

For simplicity, we will assume that the local slope varies only in the  $x$ -direction, i.e.,  $h_y = 0$ . This is really not an additional limitation—it merely simplifies the algebra. Waves may still travel in any arbitrary direction relative to the beach slope, and since the linearised equations are invariant with respect to a rotation, no generality is lost.

The dependent variables ( $\eta, u, v$ ) will have the forms:

$$\begin{aligned} \eta &= a \exp(i\phi) \\ u &= b(1 + i\sigma h_x) \exp(i\phi) \\ v &= c(1 + iv h_x) \exp(i\phi) \end{aligned} \quad (33)$$

where the frequency,  $\omega$ , cross-shore wavenumber  $k$ , and longshore wavenumber  $l$  are defined by  $\phi_t = -\omega$ ,  $\phi_x = k$ ,  $\phi_y = l$ .  $\sigma$  and  $v$  are phase shift constants, which are assumed to be of  $O(1)$ . The  $h_x$  terms are added in anticipation of the shoaling analysis to be performed later and are absent on a flat bed. The amplitudes  $a, b, c$  are assumed to be slowly varying in space. The general form of this analysis is similar to that of Schäffer and Madsen (1995b), but extended to two horizontal dimensions.

Substituting Eq. (33) into Eqs. (29) and (30) and retaining terms up to  $O(h_x)$  gives an eikonal-transport equation. The imaginary part may be solved to get the linear dispersion relationship and transfer functions between  $a, b, c$ , while the real part is used to find the evolution equation for the slowly varying wave amplitude  $a_x$  and the phase shift constants ( $\sigma, v$ ).

### 3.1. Dispersion

When Eq. (33) is substituted into Eqs. (29) and (30), the mass and momentum equations on a flat bed give rise to the matrix equation:

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (34)$$

where

$$\begin{aligned} m_{11} &= -\omega(1 - \psi_1(\kappa h)^2), & m_{12} &= (kh)\left(1 - \left(\alpha + \frac{1}{3} + \psi_1\right)(\kappa h)^2\right) \\ m_{13} &= (lh)\left(1 - \left(\alpha + \frac{1}{3} + \psi_1\right)(\kappa h)^2\right), & m_{21} &= k(1 - \rho_1(\kappa h)^2) \\ m_{22} &= \omega(-1 + (\rho_1 + \alpha)(kh)^2), & m_{23} &= \omega(\rho_1 + \alpha)(kh)(lh) \\ m_{31} &= l(1 - \rho_1(\kappa h)^2), & m_{32} &= \omega(\rho_1 + \alpha)(kh)(lh) \\ m_{33} &= \omega(-1 + (\rho_1 + \alpha)(lh)^2) \end{aligned} \quad (35)$$

where we write  $\alpha \equiv (B - 1)/2$  to emphasise the continuity with Nwogu's equations.

For nontrivial solutions the dispersion relation is, thus,

$$\frac{\omega^2}{\kappa^2 h} = \frac{1 - (\psi_1 + \rho_1 + \alpha + \frac{1}{3})(\kappa h)^2 + \rho_1(\psi_1 + \alpha + \frac{1}{3})(\kappa h)^4}{1 - (\psi_1 + \rho_1 + \alpha)(\kappa h)^2 + \psi_1(\rho_1 + \alpha)(\kappa h)^4} \quad (36)$$

where

$$\kappa = \sqrt{k^2 + l^2} \quad (37)$$

and the transfer functions between surface elevation and velocity are:

$$\begin{aligned} b &= T_{r1}a \\ c &= T_{r2}a \end{aligned} \quad (38)$$

where

$$T_{r1} = \frac{k}{\kappa} T_r$$

$$T_{r2} = \frac{l}{\kappa} T_r$$

$$T_r = \frac{\kappa(1 - \rho_1(\kappa h)^2)}{\omega(1 - (\kappa h)^2(\alpha + \rho_1))} \quad (39)$$

are the linear wave transfer functions between surface elevation and the representative velocity.

When the set 1a coefficients are used, the dispersion relationship becomes the well known Padé [2,2] approximant:

$$\frac{\omega^2}{\kappa^2 h} = \frac{1 + (\kappa h)^2/15}{1 + 2(\kappa h)^2/5} \quad (40)$$

This result was already expected, as set 1a on a flat bed is identical to Nwogu's equations with  $\alpha = -2/5$ .

For set 2, we take note of the resemblance to the MS equations, and attempt to find sets of coefficients that provide the Padé [4,4] dispersion:

$$\frac{\omega^2}{\kappa^2 h} = \frac{1 + \frac{1}{9}(\kappa h)^2 + \frac{1}{945}(\kappa h)^4}{1 + \frac{4}{9}(\kappa h)^2 + \frac{1}{63}(\kappa h)^4} \quad (41)$$

Four cases are found which give this relationship:

$$B = \frac{2}{3} - \frac{2}{63}\sqrt{133} - \frac{1}{315}\sqrt{805} \quad \text{Case [1]}$$

$$= \frac{2}{3} - \frac{2}{63}\sqrt{133} + \frac{1}{315}\sqrt{805} \quad \text{Case [2]}$$

$$= \frac{2}{3} + \frac{2}{63}\sqrt{133} - \frac{1}{315}\sqrt{805} \quad \text{Case [3]}$$

$$= \frac{2}{3} + \frac{2}{63}\sqrt{133} + \frac{1}{315}\sqrt{805} \quad \text{Case [4]}$$

$$D = \frac{135}{17}B^3 - \frac{219}{17}B^2 + \frac{793}{119}B - \frac{478}{595} \quad \text{All cases} \quad (42)$$

The values of  $(\psi_1, \rho_1)$  may be found through Eq. (28). These four sets correspond exactly to cases I–IV in MS. In fact, properties on a flat bed are identical to those in MS, while characteristics on a varying bed are much different. Armed with this foreknowledge, we shall only consider case I, and discard cases II–IV, which, in their fully nonlinear form, had nonlinear properties making them unusable. A more complete discussion of these nonlinear properties may be found in Madsen and Schäffer (1998).

Fig. 1 shows linear dispersion properties for sets 1 and 2 relative to the exact small amplitude solution:

$$\frac{\omega^2}{g\kappa} = \tanh \kappa h \quad (43)$$

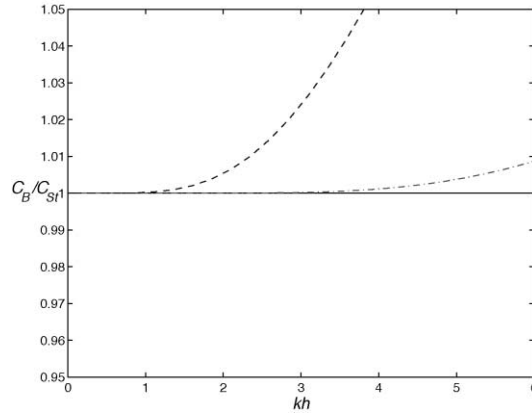


Fig. 1. Boussinesq celerities relative to exact small amplitude solution. (– –) Padé [2,2] (set 1); (– · –) Padé [4,4] (set 2).

The Padé [2,2] approximant from set 1 gives good results up to around  $\kappa h = 3$ , as is well known, but deviates considerably in deeper water. In contrast, the Padé [4,4] approximant gives much better accuracy, with good results up to around  $\kappa h = 6$ .

### 3.2. Shoaling

Once Eq. (33) is substituted into Eqs. (29) and (30), terms with slow derivatives may be grouped together and Boussinesq shoaling performance may be compared with linear Stokes theory. Discarding higher-order slope terms such as  $h_x^2$ ,  $h_{xx}$ , we arrive at a two-dimensional matrix equation that describes the two-dimensional wave shoaling rate:

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} a_x \\ \sigma \\ v \end{bmatrix} = a \begin{bmatrix} b_{11} \\ b_{22} \\ b_{33} \end{bmatrix} \tag{44}$$

where the coefficients are:

$$m_{11} = T_{r1}h \left( 1 - \left( \alpha + \frac{1}{3} + \psi_1 \right) (3(kh)^2 + (lh)^2) \right) - 2T_{r2}h(kh)(lh) \left( \alpha + \frac{1}{3} + \psi_1 \right) + 2\psi_1\omega h(kh) \tag{45}$$

$$m_{12} = h_x T_{r1}(kh) \left( -1 + \left( \alpha + \frac{1}{3} + \psi_1 \right) (\kappa h)^2 \right) \tag{46}$$

$$m_{13} = h_x T_{r2}(lh) \left( -1 + \left( \alpha + \frac{1}{3} + \psi_1 \right) (\kappa h)^2 \right) \tag{47}$$

$$m_{21} = 1 + h\omega(\alpha + \rho_1)(2T_{r1}(kh) + T_{r2}(lh)) + \rho_1(-3(kh)^2 - (lh)^2) \tag{48}$$

$$m_{22} = T_{r1}\omega h_x(1 - (\alpha + \rho_1)(kh)^2) \tag{49}$$

$$m_{23} = -T_{r2}\omega h_x(kh)(lh)(\alpha + \rho_1) \quad (50)$$

$$m_{31} = T_{r1}\omega h(lh)(\alpha + \rho_1) - 2\rho_1(kh)(lh) \quad (51)$$

$$m_{32} = -h_x T_{r1}\omega(kh)(lh)(\alpha + \rho_1) \quad (52)$$

$$m_{33} = h_x T_{r2}\omega(1 - (\alpha + \rho_1)(lh)^2) \quad (53)$$

$$\begin{aligned} b_{11} = & k_x h^2 \left( \left( \alpha + \frac{1}{3} + \psi_1 \right) (3(kh)T_{r1} + (lh)T_{r2}) - \psi_1 \omega \right) \\ & + h_x \left[ -T_{r1} \left( 1 - (kh)^2 \left( 3 \left( \alpha + \frac{1}{3} \right) + 2 \left( A - \frac{1}{2} \right) \right) - \left( A - \frac{1}{2} \right) (lh)^2 \right) \right. \\ & + T_{r2}(kh)(lh) \left( 3 \left( \alpha + \frac{1}{3} \right) + \left( A - \frac{1}{2} \right) \right) - \psi_1 (2\omega(kh) - (\kappa h)^2 T_{r1} - 4(kh)((kh)T_{r1} \\ & + (lh)T_{r2})) - \psi_2 (\omega(kh) - (kh)((kh)T_{r1} + (lh)T_{r2})) \left. \right] - T_{r1x} h \left( 1 - \left( \alpha + \frac{1}{3} + \psi_1 \right) (3(kh)^2 + (lh)^2) \right) \\ & + 2T_{r2x} h(kh)(lh) \left( \alpha + \frac{1}{3} + \psi_1 \right) \end{aligned} \quad (54)$$

$$\begin{aligned} b_{22} = & -\omega h(\alpha + \rho_1)(2T_{r1x}(kh) + T_{r2x}(lh)) - k_x h^2 \omega T_{r1}(\alpha + \rho_1) - h_x \omega (T_{r1}(kh)(2A - 2 + \rho_2 + \rho_3) \\ & + T_{r2}(lh)(A - 1 + \rho_2)) + 3k_x h(kh)\rho_1 + h_x h(\kappa^2 \rho_2 + k^2 \rho_3) \end{aligned} \quad (55)$$

$$b_{33} = -\omega h T_{r1x}(lh)(\alpha + \rho_1) - h_x \omega (lh) T_{r1}(A - 1 + \rho_3) + k_x h(lh)\rho_1 + h_x k l h \rho_3 \quad (56)$$

where

$$\begin{aligned} T_{r1x} &= T_{rx} \frac{k}{\kappa} + T_r \kappa_x \left( \frac{1}{k} - \frac{k}{\kappa^2} \right) \\ T_{r2x} &= T_{rx} \frac{l}{\kappa} - T_r \kappa_x \frac{l}{\kappa^2} \\ k_x &= \frac{\kappa}{k} \kappa_x \end{aligned} \quad (57)$$

The rate of change of wavenumber is:

$$\kappa_x = -\frac{\kappa h_x}{2h} \frac{75 - 15(\kappa h)^2 + 2(\kappa h)^4}{75 + 10(\kappa h)^2 + 2(\kappa h)^4} \quad (58)$$

for set 1 (Padé [2,2] dispersion) and:

$$\kappa_x = -\frac{\kappa h_x}{2h} \frac{59,535 - 6615(\kappa h)^2 + 420(\kappa h)^4 - 21(\kappa h)^6 + (\kappa h)^8}{59,535 + 13,230(\kappa h)^2 + 2184(\kappa h)^4 + 56(\kappa h)^6 + (\kappa h)^8} \quad (59)$$

for set 2 (Padé [4,4] dispersion). The relationship:

$$T_{rx} = \frac{\kappa_x(1 + (\alpha - 2\rho_1)(\kappa h)^2 + \rho_1(\alpha + \rho_1)(\kappa h)^4) + 2\kappa^3 h \alpha h_x}{\omega(-1 + (\kappa h)^2(\alpha + \rho_1))^2} \quad (60)$$

may be used for both sets.

The rate of change of wave amplitude is then found to be:

$$a_x = a \frac{b_{11}(m_{22}m_{33} - m_{32}m_{23}) - b_{22}(m_{12}m_{33} - m_{13}m_{32}) + b_{33}(m_{12}m_{23} - m_{22}m_{13})}{m_{11}(m_{22}m_{33} - m_{23}m_{32}) - m_{12}(m_{21}m_{33} - m_{23}m_{31}) + m_{13}(m_{21}m_{32} - m_{22}m_{31})} \quad (61)$$

For normal wave incidence ( $l=0$ ), the matrix becomes singular, but a solution may still be found for the shoaling rate:

$$a_x = a \frac{b_{11}m_{22} - b_{22}m_{12}}{m_{11}m_{22} - m_{21}m_{12}} \quad (62)$$

Before comparing these to infinite depth hyperbolic solutions, we must first derive the form of shoaling for small amplitude Stokes waves at an arbitrary angle to the bottom bathymetry, assuming a mild slope. Wave ray theory (e.g., Dean and Dalrymple, 1990) and conservation of energy give the relationship between wave heights at locations 1 and 2:

$$a_2 = a_1 \sqrt{\frac{C_{g1}}{C_{g2}}} \sqrt{\frac{d_1}{d_2}} \quad (63)$$

where  $a$  is the wave amplitude,  $C_g$  is the group velocity and  $d$  is the perpendicular distance between wave rays. Taking the differential limit as the two locations become infinitely close yields:

$$\frac{a_x}{a} = -\frac{C_{gx}}{2C_g} - \frac{d_x}{2d} \quad (64)$$

Thus, the rate of change of wave height depends on two components: the rate of change of group velocity, and the rate of change of ray density.

The first term on the right side is simply the shore-normal shoaling rate,  $-\gamma_0 h_x/h$  (e.g., Madsen and Schäffer, 1998), where:

$$\gamma_0 = \frac{2(\kappa h)\sinh 2(\kappa h) + 2(\kappa h)^2(1 - \cosh 2(\kappa h))}{(2(\kappa h) + \sinh 2(\kappa h))^2} \quad (65)$$

where, again,  $\kappa = \sqrt{(k^2 + l^2)}$  is the total wavenumber. From geometric arguments, the second term is seen to be:

$$\frac{d_x}{2d} = \frac{\kappa_x}{2\kappa} \frac{(\kappa^2 - k^2)}{k^2} \quad (66)$$

For parallel contours, the second term always tends to decrease height for oblique waves travelling into shallow water.

We may now compare the various Boussinesq solutions to the exact linear Stokes solution. To begin with, this will be done asymptotically using Taylor series (all expansions were performed using the symbolic manipulation package Maple, as expressions were extremely complex). The Boussinesq and Stokes solutions may be expanded about the long wave limit for both shore normal and oblique incidence as:

$$\frac{a_x}{a} = -\frac{h_x}{h} \gamma \quad (67)$$

where, for shore-normal waves, the Stokes Taylor series is:

$$\gamma_{St} = \frac{1}{4} - \frac{1}{4}(\kappa h)^2 + \frac{1}{18}(\kappa h)^4 + \frac{1}{540}(\kappa h)^6 - \frac{11}{3150}(\kappa h)^8 + \frac{1}{3150}(\kappa h)^{10} + O((\kappa h)^{12}) \quad (68)$$

For sets 1a and 2, the equivalent Taylor series for shore-normal waves are, respectively (evaluating complex expressions):

$$\gamma_B = \frac{1}{4} - \frac{1}{4}(\kappa h)^2 + \frac{1}{18}(\kappa h)^4 + \frac{7}{1350}(\kappa h)^6 + O(\kappa h)^8 \quad (69)$$

and

$$\begin{aligned} \gamma_B = & \frac{1}{4} - \frac{1}{4}(\kappa h)^2 + \frac{1}{18}(\kappa h)^4 + (0.028727A - 0.092014C + 0.0032977)(\kappa h)^6 + (-0.0053075A \\ & + 0.020340C - 0.0042369)(\kappa h)^8 + O((\kappa h)^{10}) \end{aligned} \quad (70)$$

For set 1a, the match is exact up to  $(\kappa h)^4$ , with errors occurring at higher order. This is not surprising, and confirms the higher-order behaviour for shore-normal waves. However, in somewhat of a surprise, analytic  $(\kappa h)^4$  behaviour also occurs for all incident wave angles (only shore normal Taylor series are shown here). Since our derivation showed that set 1a was only  $O(\mu^4)$  accurate for shore-normal waves, this good behaviour for oblique incidence is somewhat of a mystery, albeit very welcome.

For set 2, using Eqs. (28)–(42) to determine  $(B, D, \psi_1, \rho_1)$ , gives an analytic match up to  $(\kappa h)^4$  for all angles of wave incidence. However, we still have two free parameters,  $(A, C)$ , which may be used to match higher-order asymptotic coefficients. The set  $(A=0.40786, C=0.14305)$ , designated set 2a, was chosen to give the best possible asymptotic match up to  $(\kappa h)^8$  for normally incident waves. However, this set was also found to provide  $(\kappa h)^8$  accuracy for all incident wave angles, which is welcome.

If the shoaling coefficients are integrated over depth, the overall amplitude error between any initial depth and shallow water may be quantified. Such an analysis was performed by Chen and Liu (1995), who gave the relationship:

$$\frac{a_B}{a_{St}} = \exp \left[ \int_0^{h/L_0} \frac{\gamma_{St}(h'/L_0) - \gamma_B(h'/L_0)}{(h'/L_0)} d(h'/L_0) \right] \quad (71)$$

where  $L_0$  is the deep water wavelength calculated from linear theory, and subscripts *St* and *B* represent Stokes and Boussinesq quantities, respectively.

Fig. 2 shows results for sets 1a and 2a compared to the exact solution for varying initial wave angles. For all angles, set 1a gives good behaviour up to around  $h/L_0=0.3$  ( $\kappa h \approx 2$ ), but begins to deviate strongly after this, and shows just under a 10% amplitude error for waves shoaling from  $h/L_0=0.5$  ( $\kappa h \approx 3$ ). The results from Set 2a are much better with very good behaviour until around  $h/L_0=0.8$  ( $\kappa h \approx 5$ ), and a maximum error of around 5% at  $h/$

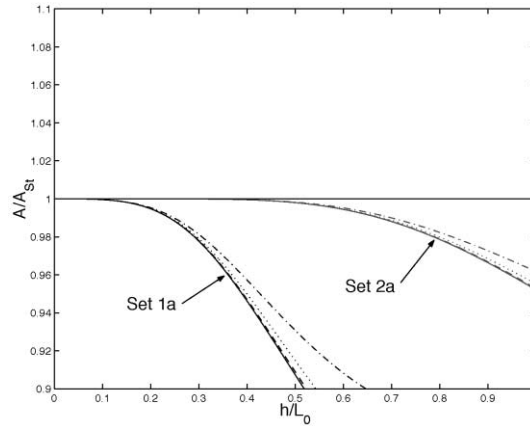


Fig. 2. Integrated shoaling amplitudes of sets 1a, 2a for deepwater angles  $0^\circ$  (—),  $15^\circ$  (---),  $30^\circ$  (···) and  $45^\circ$  (- · -). Some lines are obscured.

$L_0 = 1$  ( $\kappa h \approx 2\pi$ ). However, when comparing Figs. 1 and 2, it appears that linear dispersion has a somewhat greater range of accuracy than shoaling for both sets.

This can be remedied by considering the shoaling optimisation. For set 2, ( $A, C$ ) were chosen based on Taylor series coefficients matches up to the largest possible power,  $(\kappa h)^8$ . This guarantees good performance in shallow water, but not for large wavenumbers. If ( $A, C$ ) are instead chosen by optimising some quantity over a range, overall performance could improve. It should be emphasised that no matter what values are chosen, set 2 will always retain  $O(\mu^4)$  shoaling. For set 1, however, there are no free quantities to be changed. To modify performance here, it would be necessary to destroy  $O(\mu^4)$  behaviour on varying beds—this is what we shall do. We shall make  $A$  a free coefficient, and lose all  $O(\mu^4)$  behaviour on a varying bed for set 1. The set does retain higher-order properties on a flat bed.

To set the free coefficients, the maximum relative error:

$$RF \equiv \max\left(\frac{a_B}{a_{St}}\right) - \min\left(\frac{a_B}{a_{St}}\right), \quad 0 \leq h/L_0 \leq (h/L_0)_{\max} \tag{72}$$

was minimised for shore-normal waves. This is equivalent to minimising the maximum possible error in shoaling amplitude over a specified depth range. Based on the accuracy of the two dispersion relationships, set 1 was optimised up to  $(h/L_0)_{\max} = 0.5$  ( $\kappa h \approx 3$ ), while set 2 was optimised up to the much deeper limit of  $(h/L_0)_{\max} = 1$ .

Fig. 3 shows the final results for newly optimised sets 1b and 2b. Performance for set 1b (coefficient choice) using  $A = 0.4298$  shows RF decrease from 0.0925 to 0.0095 up to a maximum wavenumber  $h/L_0 = 0.5$  ( $\kappa h \approx 3$ ), where linear dispersion begins to lose accuracy. The largest error in this range is now less than 1%, which is a significant improvement. An even greater decrease in RF from 0.0481 to 0.0003 is visible for set 2b (enhancement) up to  $h/L_0 = 1$  ( $\kappa h \approx 2\pi$ ), using ( $A = 0.395994, C = 0.139119$ ). Accuracy also increases for oblique waves, although by somewhat less, and the usefulness of the optimisation now becomes clear.

We have also computed results for Nwogu’s equations (set 1 with  $A = 1/\sqrt{5}$ ), and for the MS equations, which are linearly identical to sets 2a and 2b on a flat bed [note that there is a typographical error in Madsen and Schäffer, 1998, p. 3159, last sentence should end with  $(\beta_2, \alpha_2) = (-0.10536, -0.01101)$ . Personal communication, H. Schäffer. MS also now recommend a revised set of coefficients after considering integrated shoaling performance,  $(\beta_2, \alpha_2) = (0.315236, 0.124537)$ . Both sets are shown and are denoted as MS (original) and MS (revised)]. However, Fig. 4 shows that shoaling performance is very much different from the present equations. When compared with the optimised results of sets 1b and 2b, Nwogu’s and the MS results are considerably worse. For set 2b

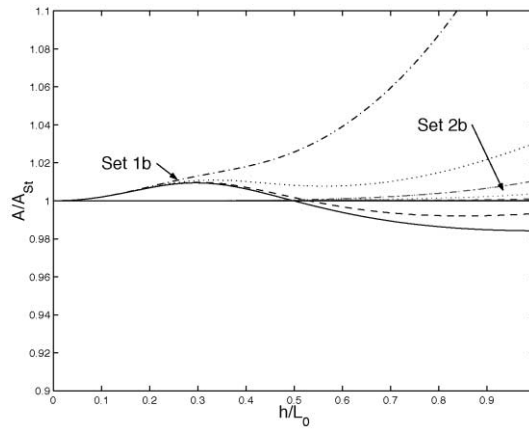


Fig. 3. Integrated shoaling amplitudes of sets 1b, 2b for deepwater angles  $0^\circ$  (—),  $15^\circ$  (---),  $30^\circ$  (···) and  $45^\circ$  (- · -). Some lines are obscured.

(enhancement), this improvement over the MS equations comes partly as a result of the  $O(\mu^4)$  behaviour, since the MS equations are only  $O(\mu^2)$  accurate on a sloping bed. The remainder of the improvement over MS (original) comes from the optimisation method. MS (original) optimised the shoaling gradient  $\gamma$  over the range  $0 < \kappa h < 6$ , while we have optimised the integral (71) over the same range. In contrast, MS (revised) optimised integrated shoaling amplitudes similarly to the present work, but using a somewhat different optimisation technique. The improvement is evident. As shown by Chen and Liu (1995), optimising the shoaling gradient only overemphasises errors in deeper water at the expense of shallow water accuracy.

Fig. 4 also shows that shoaling performance for set 1b (coefficient choice) exceeds that of Nwogu. At first glance, this is somewhat surprising, as set 1b lost  $O(\mu^4)$  accuracy on sloping beds, and is formally only as accurate as Nwogu’s equations. However, once Nwogu’s equations are optimised for dispersion, shoaling may not be manipulated. In contrast, set 1b retains one free coefficient ( $A$ ) which may be used to improve shoaling performance.

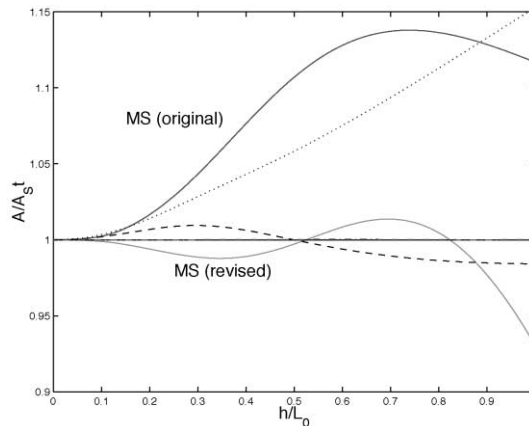


Fig. 4. Integrated shoaling amplitudes for shore-normal waves. (—) Set 1b; (---) Set 2b (obscured); (···) Nwogu; (- · -) MS (original and revised).

#### 4. Simplified very high-order equations

Just as Nwogu’s and the MS equations were shown to be of higher order than was generally believed, the nominally  $O(\mu^4)$  GK equations can also be shown to be of much higher order.

To do this, we must first derive very high-order equations, which will then be simplified using the same techniques as in the previous sections. In their full form, these equations would be extremely complex, so we will perform the derivation for the more tractable case of linear equations on a flat bed. The horizontal and vertical velocities take the form of an infinite series

$$\mathbf{u}(\mathbf{x}, z, t) = \sum_{n=0}^{\infty} \zeta^n \mathbf{u}_{(n)}(\mathbf{x}, t) \tag{73}$$

$$w(\mathbf{x}, z, t) = \sum_{n=0}^{\infty} \zeta^n w_{(n)}(\mathbf{x}, t) \tag{74}$$

where  $\zeta \equiv z + h$ .

Using the continuity Eq. (2), bottom boundary condition (3) and zero horizontal vorticity condition (4), we arrive at the Taylor series expansion for the horizontal and vertical velocities:

$$\mathbf{u}(\mathbf{x}, z) = \sum_{n=0}^{\infty} \frac{\mu^n (-1)^{n/2}}{(n)!} \zeta^n \nabla^{(n)} \mathbf{u}_{(0)}, \quad n \text{ even} \tag{75}$$

$$w(\mathbf{x}, z) = \sum_{n=2}^{\infty} \frac{\mu^n (-1)^{n/2}}{(n-1)!} \zeta^{n-1} \nabla^{(n-1)} \mathbf{u}_{(0)}, \quad n \text{ even} \tag{76}$$

where we define  $\nabla^{(1)} \equiv \nabla \cdot$ ,  $\nabla^{(2)} \equiv \nabla \nabla \cdot$ , etc., and  $\mathbf{u}_{(0)}$  is the bed velocity.

These well-known series have been presented by numerous authors. Similar to Wu (1999), it would be straightforward to substitute these relations into the conservation of mass and momentum equations and arrive at infinite-order Boussinesq equations in terms of the bottom velocity,  $\mathbf{u}_{(0)}$ . However, we wish to

make the equations very general, so that their most useful forms may be found.

To do this, we use the concept of Schäffer and Madsen (1995a,b), and introduce an yet unspecified function  $f(\zeta)$ . With some definitions:

$$A^{(0)} h^0 = 1 \equiv \frac{\int_0^h f(\zeta) d\zeta}{\int_0^h f(\zeta) d\zeta}$$

$$A^{(1)} h = Ah \equiv \frac{\int_0^h f(\zeta) \zeta d\zeta}{\int_0^h f(\zeta) d\zeta}$$

$$A^{(2)} h^2 = Bh^2 \equiv \frac{\int_0^h f(\zeta) \zeta^2 d\zeta}{\int_0^h f(\zeta) d\zeta}$$

$$A^{(3)} h^3 = Ch^3 \equiv \frac{\int_0^h f(\zeta) \zeta^3 d\zeta}{\int_0^h f(\zeta) d\zeta}$$

...

$$\bar{\mathbf{u}} \equiv \frac{\int_0^h f(\zeta) \mathbf{u}(\zeta) d\zeta}{\int_0^h f(\zeta) d\zeta} \tag{77}$$

we may generalise the velocity variable  $\bar{\mathbf{u}}$  and, thus, the resulting equations. In the general case, the function  $f(\zeta)$  could vary in both space and time (e.g., Schäffer and Madsen, 1995a). However, for the flat bed, linear problem, we will assume that  $f(\zeta)$  is constant. This means that the  $A^{(n)}$  coefficients are also constant, and their relationship to  $A, B, C, D$ , etc., becomes clear. In practice, it is simplest to

specify directly the various  $A^{(n)}$ 's and, thus, a full description of  $f(\zeta)$  is rarely needed. Indeed, at any finite truncated level there are usually an infinite number of functions which will yield the same integrals and, thus, the same equations.

The generalised velocity (Eq. (77)) may be introduced at one of two points: (1) after first inverting Eqs. (75) and (76) in terms of a single  $z_\alpha$  reference elevation, as in Schäffer and Madsen (1995a), or (2) before inversion, as in GK. The two methods give somewhat different equations of apparently similar accuracy. As we wish to examine the GK equations, we will employ method (2) (it should be possible to use both methods and create even more generalised equations; however, it is not clear that these would offer any significant advantages in performance).

Omitting the details of the derivation, which uses standard techniques, and keeping terms up to  $O(\mu^8)$ , velocities are:

$$\begin{aligned}
 \mathbf{u}(\mathbf{x}, z) = & \tilde{\mathbf{u}} + \frac{\mu^2}{2} [Bh^2 - \zeta^2] \nabla \nabla \cdot \tilde{\mathbf{u}} \\
 & + \frac{\mu^4}{4} \left[ \left( B^2 - \frac{1}{6} D \right) h^4 - Bh^2 \zeta^2 \right. \\
 & \left. + \frac{1}{6} \zeta^4 \right] \nabla \nabla^2 \nabla \cdot \tilde{\mathbf{u}} \\
 & + \frac{\mu^6}{8} \left[ \left( B^3 - \frac{1}{3} BD + \frac{1}{90} F \right) h^6 \right. \\
 & \left. - \left( B^2 - \frac{1}{6} D \right) h^4 \zeta^2 + \frac{1}{6} Bh^2 \zeta^4 \right. \\
 & \left. - \frac{1}{90} \zeta^6 \right] \nabla \nabla^2 \nabla^2 \nabla \cdot \tilde{\mathbf{u}} \\
 & + \frac{\mu^8}{16} \left[ \left( B^4 - \frac{1}{2} B^2 D + \frac{1}{45} BF \right) \right. \\
 & \left. + \frac{1}{36} D^2 - \frac{1}{2520} H \right] h^8 \\
 & - \left( B^3 - \frac{1}{3} BD + \frac{1}{90} F \right) h^6 \zeta^2 \\
 & + \frac{1}{6} \left( B^2 - \frac{1}{6} D \right) h^4 \zeta^4 \\
 & - \frac{1}{90} Bh^2 \zeta^6 + \frac{1}{2520} \zeta^8 \left. \right] \nabla \nabla^2 \nabla^2 \nabla^2 \nabla \cdot \tilde{\mathbf{u}} + O(\mu^{10}) \tag{78}
 \end{aligned}$$

$$\begin{aligned}
 w(\mathbf{x}, z) = & -\mu^2 \zeta \nabla \cdot \tilde{\mathbf{u}} + \frac{\mu^4}{2} \left( -B\zeta + \frac{1}{3} \zeta^3 \right) \nabla^2 \nabla \cdot \tilde{\mathbf{u}} \\
 & + \frac{\mu^6}{4} \left( \left( -B^2 + \frac{1}{6} D \right) \zeta \right. \\
 & \left. + \frac{1}{3} B\zeta^3 - \frac{1}{30} \zeta^5 \right) \nabla^2 \nabla^2 \nabla \cdot \tilde{\mathbf{u}} \\
 & + \frac{\mu^8}{8} \left( \left( -B^3 + \frac{1}{3} BD - \frac{1}{90} F \right) \zeta \right. \\
 & \left. + \left( \frac{1}{3} B^2 - \frac{1}{18} D \right) \zeta^3 - \frac{1}{30} B\zeta^5 \right. \\
 & \left. + \frac{1}{630} \zeta^7 \right) \nabla^2 \nabla^2 \nabla^2 \nabla \cdot \tilde{\mathbf{u}} + O(\mu^{10}) \tag{79}
 \end{aligned}$$

The linearised mass and momentum equations become:

$$\begin{aligned}
 \eta_t + h \nabla \cdot \tilde{\mathbf{u}} + \frac{\mu^2}{2} \left( B - \frac{1}{3} \right) h^3 \nabla^2 \nabla \cdot \tilde{\mathbf{u}} \\
 + \frac{\mu^4}{4} \left( B^2 - \frac{1}{3} B - \frac{1}{6} D + \frac{1}{30} \right) h^5 \nabla^2 \nabla^2 \nabla \cdot \tilde{\mathbf{u}} \\
 + \frac{\mu^6}{8} \left( B^3 - \frac{1}{3} B^2 + \frac{1}{30} B - \frac{1}{3} BD + \frac{1}{18} D \right. \\
 \left. + \frac{1}{90} F - \frac{1}{630} \right) h^7 \nabla^2 \nabla^2 \nabla^2 \nabla \cdot \tilde{\mathbf{u}} \\
 + \frac{\mu^8}{16} \left( B^4 - \frac{1}{3} B^3 + \frac{1}{30} B^2 - \frac{1}{630} B - \frac{1}{2} B^2 D \right. \\
 \left. + \frac{1}{45} BF + \frac{1}{9} BD + \frac{1}{36} D^2 - \frac{1}{180} D - \frac{1}{270} F \right. \\
 \left. - \frac{1}{2520} H + \frac{1}{22680} \right) h^9 \nabla^2 \nabla^2 \nabla^2 \nabla^2 \nabla \cdot \tilde{\mathbf{u}} = 0 \tag{80}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\mathbf{u}}_t + \nabla \eta + \frac{\mu^2}{2} (B - 1) h^2 \nabla \nabla \cdot \tilde{\mathbf{u}}_t \\
 + \frac{\mu^4}{4} \left( B^2 - B - \frac{1}{6} D + \frac{1}{6} \right) h^4 \nabla \nabla^2 \nabla \cdot \tilde{\mathbf{u}}_t \\
 + \frac{\mu^6}{8} \left( B^3 - B^2 + \frac{1}{6} B - \frac{1}{3} BD + \frac{1}{6} D \right. \\
 \left. + \frac{1}{90} F - \frac{1}{90} \right) h^6 \nabla \nabla^2 \nabla^2 \nabla \cdot \tilde{\mathbf{u}}_t
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\mu^8}{16} \left( B^4 - B^3 + \frac{1}{6} B^2 - \frac{1}{90} B - \frac{1}{2} B^2 D \right. \\
& + \frac{1}{3} B D + \frac{1}{45} B F + \frac{1}{36} D^2 - \frac{1}{36} D - \frac{1}{90} F \\
& \left. - \frac{1}{2520} H + \frac{1}{2520} \right) h^8 \nabla \nabla^2 \nabla^2 \nabla^2 \nabla \cdot \tilde{\mathbf{u}}_t = 0
\end{aligned} \tag{81}$$

These equations are very complex and are impractical for use. However, they, too, can be simplified in much the same manner as previously. The first simplification technique was through coefficient choice. If the set ( $B=1/9$ ,  $D=5/189$ ,  $F=1/81$ ,  $H=-65/5103$ ) is chosen, all  $O(\mu^6)$  and  $O(\mu^8)$  terms disappear from Eqs. (80) and (81). The resulting equations are the GK equations on a flat bed using the set ( $B=1/9$ ,  $D=5/189$ ), which gives an excellent Padé [4,4] dispersion relationship. Thus, the GK equations are in this situation actually linearly  $O(\mu^8)$  accurate on a flat bed, even though they only show  $O(\mu^4)$  terms. Note that because the even coefficient  $H$  is negative, ( $H=-65/5103$ ), the function  $f(\zeta)$  must be complex.

Based on previous experience with set 1, it appears likely that, with a proper choice of ( $A$ ,  $C$ ), the GK equations could also be made to be linearly  $O(\mu^8)$  accurate on slopes of  $O(\nabla h)$ . This would probably only be true for waves parallel to the bottom contours, as was found for set 1 (coefficient choice) of the simplified  $O(\mu^4)$  equations. However, due to the extreme complexity of the equations on varying topography, such an analysis will not be attempted.

It is also possible to simplify the  $O(\mu^8)$  equations using an enhancement technique. Here, we apply the differential operators  $1 + \psi_1 \mu^2 \nabla \cdot (h^2 \nabla)$  and  $1 + \mu^2 \rho_1 h^2 \nabla (\nabla \cdot)$  to the mass and momentum equations (Eqs. (80) and (81)). We now attempt to simplify the equations by cancelling all  $O(\mu^8)$ ,  $O(\mu^6)$ , and  $O(\mu^4)$  terms linearly on a flat bed. Solving the complex algebraic equations with the assistance of Maple, we arrive at four solutions which exactly correspond to cases I–IV of set 2 (Eq. (42)). Case I is of most interest to us because of the superior properties of its equivalent nonlinear equations (Madsen and Schäffer, 1998), and gives the set ( $B=0.2105$ ,  $D=0.1026$ ,  $F=0.06826$ ,  $H=0.03992$ ,  $\psi_1=-0.03917$ ,  $\rho_1=-0.01052$ ). Linear dispersion gives the excellent Padé [4,4] relationship shown earlier.

Therefore, set 2 of the simplified  $O(\mu^4)$  equations are actually simplified  $O(\mu^8)$  equations linearly on a flat bed, only showing  $O(\mu^2)$  terms. Since the MS equations are identical to set 2 linearly on a flat bed, they are also  $O(\mu^8)$  in this case, even though they were derived as  $O(\mu^2)$  equations.

All of this begs the question—is any set of equations which has linear dispersion that is  $(\kappa h)^n$  accurate actually a  $O(\mu^n)$  set of equations linearly on a flat bed? We have not proved this, but examination of all available evidence does not disprove the supposition. Nwogu's equations with Padé [2,2] dispersion ( $(\kappa h)^4$  accurate) are actually  $O(\mu^4)$  equations, and both set 2 (MS) and the GK equations with Padé [4,4] dispersion [ $(\kappa h)^8$  accurate] are actually  $O(\mu^8)$  equations on a level bed. Madsen and Schäffer (1998) give additional positive examples, and we have been unable to find any contradictory sets. Thus, in common with Madsen and Schäffer (1998) we take as a working guide that any set of equations which has linear dispersion accurate to  $(\kappa h)^n$ , is actually  $O(\mu^n)$  accurate linearly on a flat bed.

On a sloping bed, the answer is less clear: set 1 had shoaling rates that were  $(\kappa h)^4$  accurate for all incident wave angles, but was formally of  $O(\mu^4)$  only for waves parallel to bottom contours. In contrast, set 2 also had  $(\kappa h)^4$  accurate shoaling, but was completely  $O(\mu^4)$  on slopes of  $O(\nabla h)$ . In between are shoaling analyses (such as Schäffer and Madsen, 1995a combined with Madsen and Schäffer, 1998,  $U$  formulation) which show both  $(\kappa h)^4$  accuracy and  $O(\mu^4)$  analyticity but only consider one horizontal dimension. Due to the complexities of shoaling analyses, few others have been performed in one dimension and, except for this paper, we know of none in two horizontal dimensions. Nevertheless, we know of no analyses for waves parallel to bottom contours (one dimension) where  $(\kappa h)^n$  accurate shoaling has not led to  $O(\mu^n)$  analytic accuracy on slopes of  $O(h_x)$ .

## 5. Internal velocities

For future applications such as sediment transport modelling, representation of internal fluid velocities will likely be important. Use of the higher-order velocity profiles associated with the simplified higher-order equations can improve predictions significantly, although they should not be used blindly.

On a flat bed, all interior velocity profiles up to  $O(\mu^8)$  may be represented by Eqs. (78) and (79). For each of our various sets of equations, the Boussinesq velocities may, thus, be compared to exact hyperbolic solutions for a steady linear wave simply by choosing the appropriate coefficients ( $B, D, F, H$ ), and keeping terms up to the appropriate level of approximation. For simplicity, we will show only horizontal velocities, as vertical velocities show identical trends in terms of accuracy and convergence.

Fig. 5 shows horizontal velocity profiles for sets 1a and 1b, which are identical on a flat bed. Velocities are normalised to match at the still water level. Results are shown at the  $O(\mu^2)$  level (Nwogu’s result) and at  $O(\mu^4)$ . At  $O(\mu^2)$ , velocities begin to diverge from the exact solution fairly quickly after  $\kappa h = 3$ , and horizontal bottom velocities even become negative for high wavenumbers.  $O(\mu^4)$  fluid velocities for set 1 look much better at small wavenumbers, but diverge even more strongly than the lower-order results for high wavenumbers, where they are very poor. However, it should be noted that dispersion for these sets is only accurate up to  $\kappa h = 3$ , where interior velocities still show reasonable results.

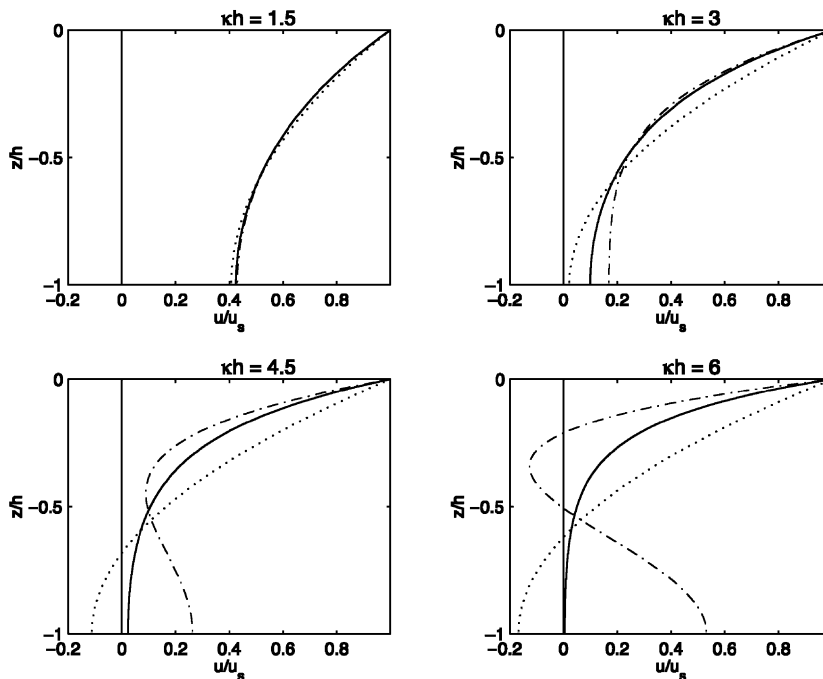


Fig. 5. Horizontal fluid velocities under a small-amplitude wave. (—) Exact; (· · ·) set 1,  $O(\mu^2)$  profile; (— · —) Set 1,  $O(\mu^4)$ .

Fig. 6 shows results for set 2 at  $O(\mu^2)$  (MS assumed level),  $O(\mu^4)$  (as derived), and  $O(\mu^8)$  (as demonstrated in the previous section). Lower-order results again begin to diverge at fairly small wavenumbers, while  $O(\mu^4)$  results show significantly better agreement up to  $\kappa h = 3$  but diverge strongly from the exact solution for  $\kappa h > 3$ . In a large disappointment,  $O(\mu^8)$  velocity profiles give excellent results for  $\kappa h \leq 3$ , but diverge very strongly at high wavenumbers.

Finally, the unsimplified  $O(\mu^4)$  GK equations show reasonably good agreement with exact fluid velocities up to  $\kappa h = 6$  as seen in Fig. 7. This is particularly good when compared with the poor high wavenumber performance of the sets 1 and 2 at  $O(\mu^4)$ . At  $O(\mu^8)$ , the GK equations also show good behaviour, with very small errors at  $\kappa h = 4.5$ , although some error is visible at  $\kappa h = 6$ . Still, overall behaviour must be considered the best of all sets of equations.

It, thus, becomes clear that interpreting sets 1 and 2 as  $O(\mu^4)$  equations can give good results at low wavenumbers, but is somewhat perilous for very short waves. Interpreting set 2 as  $O(\mu^8)$  equations can give particularly poor results for large wavenumbers. The GK equations generally give quite good results, no

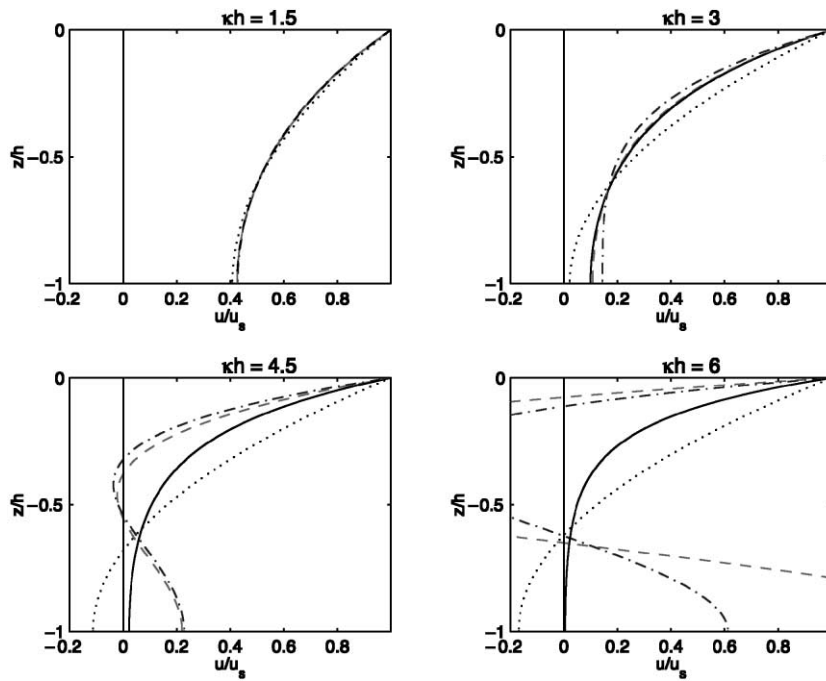


Fig. 6. Horizontal fluid velocities under a small-amplitude wave. (—) Exact; (· · ·) set 2,  $O(\mu^2)$  profile; (- · -) set 2,  $O(\mu^4)$ ; (- -) Set 2,  $O(\mu^8)$ .

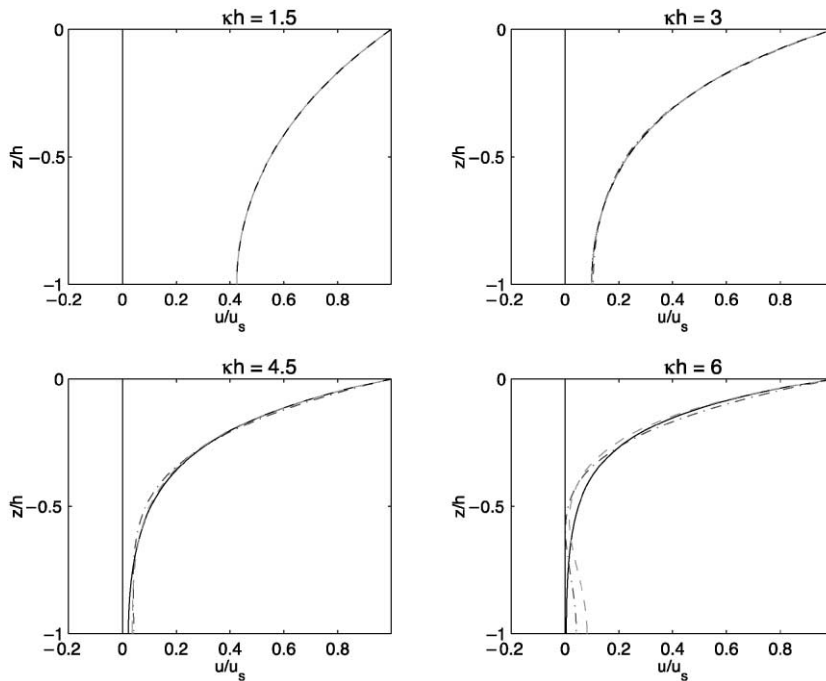


Fig. 7. Horizontal fluid velocities under a small-amplitude wave. (—) Exact; (- · -)  $O(\mu^4)$  GK equations; (- -)  $O(\mu^8)$  GK equations.

matter whether they are interpreted as  $O(\mu^4)$  or  $O(\mu^8)$  equations.

The reasons for this loss of accuracy at large wavenumbers are initially somewhat puzzling. Loss of accuracy for set 1 (coefficient choice) might be expected for large wavenumbers, as linear dispersion gives good results only until  $\kappa h = 3$ . However, set 2 (enhancement) is  $O(\mu^8)$  accurate on a flat bed, and was expected to give much better accuracy. Small wavenumbers do give significantly better results, but it appears that the Taylor series expansion is not sufficient to guarantee accuracy for large wavenumbers.

With some work, it becomes apparent that the poor performance is due to lack of convergence in the Taylor series inversion. As noted by Wu (1999), the original series (Eq. (75)) has an infinite radius of convergence, but “inverted series are known to possess a finite range of convergence.” Because moments  $A^{(n)}$  are not known to infinity, details become difficult; however, it is possible to get an analytic radius of convergence if a single  $z_x$  is used ( $f(\zeta) = \delta(z_x)$ ), where  $\delta$  is the Dirac delta function.

The inverted series at the bed may be written analytically as, after correcting typos in Wu (1999),

$$\mathbf{u}_{(0)} = \sum_{n=0}^{\infty} \mu^n \frac{(-1)^{n/2}}{n!} E_{(n)} (A^{(1)}h)^n \nabla^{(n)} \mathbf{u}, \quad n \text{ even} \tag{82}$$

where  $E_{(n)}$  are the Euler numbers. Substituting in an approximation for  $E_{(n)}$  valid for large  $n$  (Abramowitz and Stegun, 1964, Eq. (23.1.15)):

$$E_{(n)} \approx (-1)^{n/2} \frac{2^{n+2}n!}{\pi^{n+1}}, \quad n \text{ even} \tag{83}$$

the convergence criterion for the bed velocity  $\mathbf{u}_{(0)}$  in terms of the reference velocity,  $\bar{\mathbf{u}}$ , then becomes:

$$\kappa h < \frac{\pi}{2A^{(1)}} \tag{84}$$

This expression is strictly valid only for the restricted case ( $f(\zeta) = \delta(\zeta)$ ). For Nwogu’s equations with Pade [2,2] dispersion (Eq. (84)) is exact, and predicts that interior velocities converge for  $\kappa h < 3.5$ . For situations where there are multiple or generalised

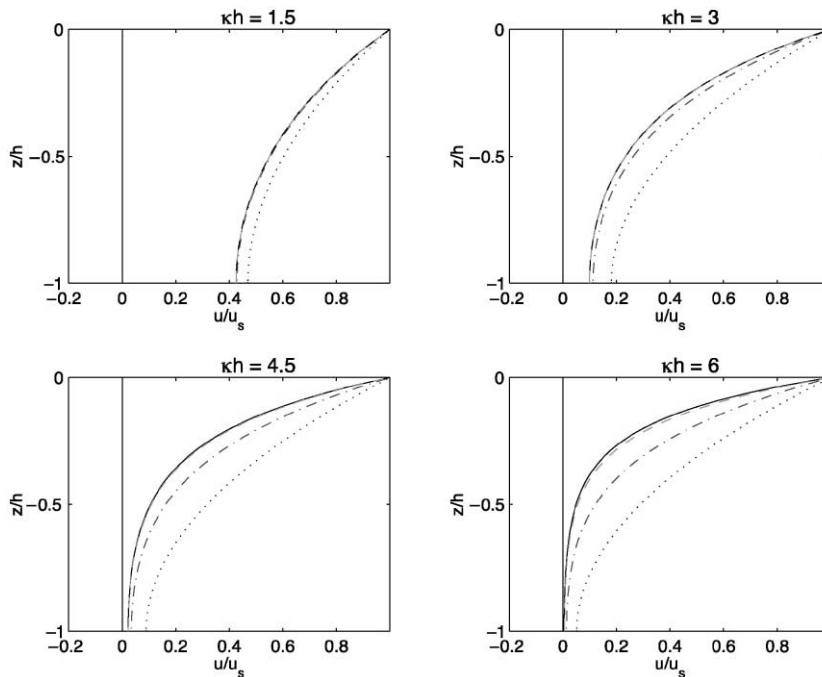


Fig. 8. Horizontal fluid velocities under a small-amplitude wave computed using bottom velocity  $\mathbf{u}_{(0)}$ . (—) Exact; (···)  $O(\mu^2)$ ; (- · -)  $O(\mu^4)$ ; (- -)  $O(\mu^8)$ .

reference elevations, Eq. (84) also holds if the generalised velocity is introduced after first inverting Eqs. (75) and (76) in terms of a single reference elevation (e.g., Schäffer and Madsen, 1995a). In contrast, if a generalised velocity is introduced before inversion (e.g., GK), then we have no exact counterpart to Eq. (84). However, as a general rule, the coefficients ( $A$ ,  $B$ ,  $C$ ,  $D$ ), etc., should be as small as possible ( $f(\zeta)$  should be weighted close to the bed) to ensure that interior velocities converge.

This explains the reasonably good performance of the GK equations, which have coefficients for Padé [4,4] dispersion ( $B=1/9$ ,  $D=5/189$ ). These are smaller than the equivalent Nwogu (or set 1) coefficients ( $B=1/5$ ,  $D=1/25$ ) and thus, the GK velocities in Fig. 7 converge for larger wavenumbers than the set 1 velocities in Fig. 5. The set 2 coefficients ( $B=0.2105$ ,  $D=0.1026$ ) are even larger than those in set 1 and, thus, convergence is worse. Even  $O(\mu^8)$  velocity profiles shown in Fig. 6 do poorly in deeper water, as the radius of convergence has been surpassed.

To further illustrate convergence and divergence, Fig. 8 shows interior fluid velocities computed using the bottom velocity  $\mathbf{u}_{(0)}$  ( $A^{(0)}=1$ ,  $A^{(n)}=0$ ,  $n=1, 2, \dots$ ). Agreement improves steadily as more terms are added to the series, and  $O(\mu^8)$  accuracy is excellent. This is not surprising since, according to Eq. (84), the radius of convergence is infinite. However, simplified  $O(\mu^4)$  equations based on the bottom velocity  $\mathbf{u}_{(0)}$  show a singularity in the linear dispersion relationship at  $\kappa h \approx 4.5$ , making them unusable. Similar problems were found for a variety of other simplified equations with small  $A^{(n)}$  values. However, for unsimplified  $O(\mu^4)$  equations, such as GK, it appears relatively straightforward to obtain good accuracy in both dispersion and internal velocities.

In somewhat of a surprise it, thus, appears that set 1, with dispersion and internal velocities good up to  $\kappa h=3$ , provides slightly better overall performance than set 2 using these criteria. The original GK equations are even more accurate, although more complex.

## 6. Conclusions

The concept of creating higher-order Boussinesq equations that only show lower-order terms has been explored linearly. Provided that the starting equations

have sufficient generality, it is quite straightforward to nullify higher-order terms using one of two techniques—coefficient choice or enhancement.

Using coefficient choice (set 1a), linear behaviour is  $O(\mu^4)$  on a flat bed, while on a sloping bed of  $O(\nabla h)$ , equations are formally  $O(\mu^4)$  only for waves parallel to the bottom contours. On a flat bed, these are identical to Nwogu's (1993) equations with  $\alpha = -2/5$ . A shoaling analysis showed that errors in shoaling performance increased more quickly with increasing wavenumber than errors in the Padé [2,2] dispersion. This led to another, preferred, set (1b) which does not have any  $O(\mu^4)$  properties on sloping beds, but has smaller shoaling errors at higher wavenumbers.

Using enhancement (set 2),  $O(\mu^4)$  behaviour was found for both flat beds and slopes of  $O(\nabla h)$ . Furthermore, four free coefficients remained which were used to optimise linear dispersion and shoaling. Dispersion was chosen to give a Padé [4,4] relationship, while two methods were used to optimise shoaling. One (set 2a) matched Taylor series coefficients with exact linear Stokes solutions, while a second set (2b) was optimised over a range. For general use, set 2b was found to have somewhat greater shoaling accuracy in deeper water. On a flat bed with Padé [4,4] dispersion, these equations were found to be identical to the MS equations and to those of Zou (2000).

Generalised equations were derived to infinite order linearly on a flat bed, and were given explicitly up to  $O(\mu^8)$ . These were shown to reduce to the GK equations (Gobbi and Kirby, 1999) in a special case, showing that the GK equations with Padé [4,4] dispersion are actually linearly  $O(\mu^8)$  accurate on flat beds, even though they were only derived as  $O(\mu^4)$  equations. Using enhancement, these equations were also shown to reduce to set 2 (or MS), resulting in equations that were linearly accurate to  $O(\mu^8)$  on a flat bed, even though they only show  $O(\mu^2)$  terms.

However, even though very good results could be obtained for dispersion and shoaling, interior velocities were not represented well at high wavenumbers. This was found to be due to a finite radius of convergence of inverted Taylor series. The inaccuracies are such that the representation of interior velocities for  $\kappa h > 3$  is not recommended using either set 1 or 2. For set 1, this is not a large restriction, as dispersion is valid only until  $\kappa h=3$ . However, set 2 has dispersion and shoaling accurate to  $\kappa h=6$ , so the restriction on interior veloc-

ities  $\kappa h < 3$  is much more limiting. Interior velocities for both sets 1 and 2 were considerably worse than those of the GK equations, which were reasonable up to near  $\kappa h = 6$ .

The question now arises as to which set (1b or 2b) should be used. Set 2b (enhancement) has obvious advantages as linear dispersion and shoaling are accurate up to  $\kappa h = 6$ , while set 1b (coefficient choice) only has accuracy up to  $\kappa h = 3$ . However, set 2b is nontrivially more complex than set 1b, although not prohibitively so. Furthermore, set 1b can represent internal velocities up to the limit of dispersion, while set 2b has somewhat worse representation of internal velocities. Dodging the question somewhat, we feel that the appropriate set depends on the problem studied. For the numerous studies where the deep water linear accuracy of set 2b is not necessary, set 1b will give very good results with less complexity, and can provide reasonable estimates of internal velocities within its range of validity. However, if shoaling from very deep water is to be simulated, set 2b provides unparalleled accuracy with a reasonable computational cost. However, it should not be used to simulate internal velocities, except in shallow water. For studies where both wave transformation and internal velocities are important in relatively deep water, unsimplified  $O(\mu^4)$  equations such as the GK equations are required, although these will increase complexity and computational expense considerably.

Most computations using Boussinesq equations are likely to feature nonlinear waves; however, the work here has dealt exclusively with linear simplifications and linear properties. Deriving nonlinear Boussinesq equations that retain higher-order linear properties is reasonably straightforward. Since sets 1 and 2 of the new equations closely resemble well-known lower-order equations (Nwogu, WKGS, and MS), it is most easily accomplished by deriving the corresponding modified nonlinear  $O(\mu^2)$  equations and choosing coefficients such that they linearly become higher order. Nonlinear computations using coefficients slightly different from sets 1b and 2b may be found in Kennedy et al. (2000b) (see also Gobbi et al., 2000b). These also provide computational examples of wave evolution over topography and give demonstrations of practical accuracy.

It should also be possible to modify the techniques used here (coefficient choice, enhancement) to create

equations which are nonlinearly of higher order, while only showing lower-order terms. Nonlinearity would probably not be fully higher order—it would likely be necessary to confine analysis to weakly nonlinear properties, and possibly only for important limited cases (such as self-interactions). It also remains to be seen if the gain in accuracy would be enough to offset the likely increase in complexity. Investigations continue and are to be reported as part 2 when complete.

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