Improved Performance in Boussinesq-type Equations

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Abstract

In this paper, simple but effective techniques are used to improve the performance of two commonly-used sets of Boussinesq-type equations. The $O(\mu^2)$ *equations of Wei et al. (1995) are extended through a redefinition of the reference velocity variable, which allows both linear shoaling and nonlinear properties to be improved significantly. Computational examples of nonlinear shoaling waves demonstrating this improvement are presented.*

The enhanced equations of Madsen and Schäffer (1998) are improved through a comparison with the higher-order Boussinesq equations of Gobbi and Kirby (1999). Through the selective cancellation of higher-order terms, equations result that are linearly of $O(\mu^4)$ *for a flat bed and for mild slopes in two dimensions. Properties on a flat bed are identical to Madsen and Schäffer (1998), but performance on a sloping bed is significantly improved.*

Introduction

In recent years, Boussinesq-type equations have become the system of choice for representing nonlinear wave transformation and wave-induced currents in the nearshore. Basic improvements to Boussinesq formulations, extensions which allow wave breaking and runup to be computed, and increased computational capabilities have led to models which can simulate a wide range of nearshore phenomena. These include wave transformation over complex topographies wave breaking and runup and wave-induced nearshore currents (Madsen et al., 1997; Chen et al., 1999; Kennedy et al., 2000a; Chen et al., 2000). A particular strength of Boussinesq equations is the natural representation of nonlinearity offered by the time domain formulation.

This paper deals with basic improvements in Boussinesq formulations, which has been a topic of considerable interest for the last decade. Beginning with papers such as Madsen and Sørensen (1992), which improved linear dispersion and shoaling properties in a flux-type Boussinesq model, there has been much effort aimed at improving

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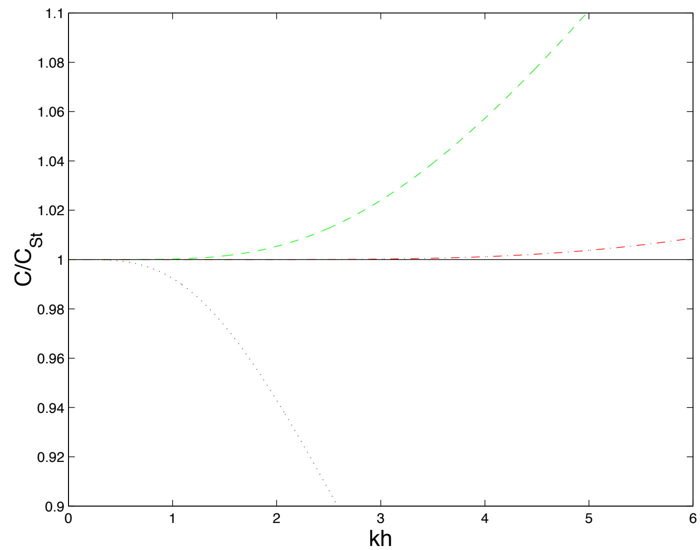

Figure 1: Boussinesq celerities compared to linear Stokes solution. $(\cdots)$ Peregrine (1967), Padé [0,2]; $(-\ -)$ Madsen and Sørensen, Nwogu, WKGS, Padé [2,2]; $(-\cdot-)$ Madsen and Schäffer, Gobbi, Padé [4,4]

basic irrotational properties of Boussinesq equations. Nwogu (1993) defined the representative velocity variable to be that at an arbitrary elevation and found that this gave a similar improvement in dispersion. Wei et al. (1995, herein denoted WKGS) extended Nwogu's work by retaining all nonlinear terms at $O(\mu^2)$ and found significant improvements for highly nonlinear shoaling waves. These equations have found acceptance for a variety of situations. Madsen and Schäffer (1998, Section 6, MS) consistently manipulated the WKGS equations using differential operators with free coefficients and arrived at somewhat more complex equations with much improved linear dispersive and shoaling properties. These equations are also widely used in coastal engineering practice. Figure 1 shows the various dispersion relationships compared to the exact small amplitude dispersion relationship, and to the "classical" Boussinesq equations of Peregrine (1967). The dispersion found in Madsen and Sørensen (1993) and Nwogu (1993) is a Padé [2,2] approximant to the exact small amplitude dispersion relationship, while MS results in Padé [4,4] accuracy. The additional accuracy resulting from the Padé [4,4] relationship is very clear.

However, neither the WKGS nor the MS equations are perfect. The WKGS equations can overpredict crest elevations just before breaking (Wei et al., 1995, Kennedy et al., 2000a) and the range of validity for linear shoaling is less than the dispersive range. The MS equations, although they have some properties which are valid into quite deep water, are formally of $O(\mu^2)$, and thus some questions remain over results in deeper water. These questions will be addressed here. The WKGS equations are to be improved by a redefinition of the velocity variable, while the MS equations will be reworked after comparison with the higher-order Boussinesq equations of Gobbi and

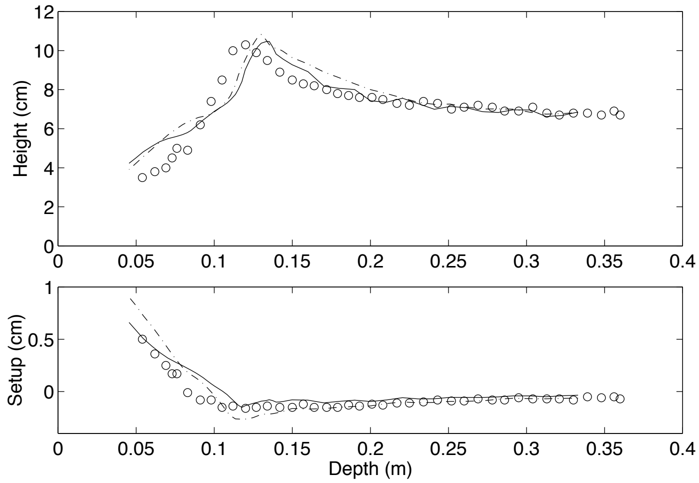

Figure 2: Computed and measured wave heights and setup for Hansen and Svendsen spilling breaker 061071; data (o); WKGS (–); Nwogu (– · –) (from Kennedy et al., 2000a)

Kirby (1999).

Improving the WKGS Equations: Highly Accurate $O(\mu^2)$ equations

The $O(\mu^2)$ WKGS equations have been shown to provide good accuracy for a wide variety of problems (Chen et al., 1999, 2000). However, although we would like to use them to compute "shelf to shore" wave transformation, they have certain properties which show greater relative error than the Padé [2,2] linear dispersion. One example of this is the second harmonic of a steady Stokes wave, which has approximately a 45% amplitude error at the deep water limit of $kh = 3$. In addition to this asymptotic nonlinear error in intermediate and deep water, the WKGS equations also show some finite amplitude error in shallow water. Figure 2, taken from Kennedy et al. (2000a), shows a comparison between computations using the WKGS and Nwogu's equations, and the experimental breaking wave measurements of Hansen and Svendsen (1979). The point of interest here is that the WKGS equations overpredict wave height just before breaking, causing an error in the breakpoint location. This is an intrinsic property of the WKGS equations and is unrelated to any inaccuracies in the breaking scheme.

To combat this error, the representative velocity used in the WKGS equations was generalised to allow independent manipulation of linear dispersion, linear shoaling and nonlinear properties. In the WKGS equations, the representative velocity, $\mathbf{u}_\alpha$(first introduced by Nwogu , 1993), is taken to be the velocity at some elevation $z_\alpha(x, y) = \zeta h(x, y)$. The free parameter, ζ, is chosen to optimise linear dispersion, which for a Padé [2,2] approximant gives $\zeta = 1/\sqrt{5} - 1$. In the new equations introduced here, the reference velocity, $\mathbf{u}_\alpha$ is composed of two components, $\mathbf{u}_\alpha = (\mathbf{u}_{\alpha 1} + \mathbf{u}_{\alpha 2})/2$, at reference elevations $z_{\alpha 1}(x, y) = \zeta_1 h(x, y) + \beta_1 \eta(x, y)$ and $z_{\alpha 2}(x, y) = \zeta_2 h(x, y) + \beta_2 \eta(x, y)$. Initially, the parameters $\zeta_1, \zeta_2, \beta_1, \beta_2$ are unknown, but will be specified based on Boussinesq approximations of exact analytic solutions for special cases. Using two reference elevations will allow us to manipulate independently linear dispersion and linear shoaling, while including a portion of the reference elevation that moves with the free surface admits the possibility of modifying nonlinear properties.

After two further definitions, $hA \equiv \frac{1}{2}\left((z_{\alpha 1}+h)+(z_{\alpha 2}+h)\right)$, $h^2B \equiv \frac{1}{2}\left((z_{\alpha 1}+h)^2+(z_{\alpha 2}+h)^2\right)$, and skipping the details of the derivation, we may now write down the modified equations

$$\eta_t + \nabla\cdot \mathbf{M} = 0 \tag{1}$$

$$\begin{aligned}\mathbf{M} = (h+\delta\eta)\Bigg\{\mathbf{u}_\alpha &+ \mu^2\left[Ah - \frac{(h+\delta\eta)}{2}\right]\left(\nabla(\nabla\cdot(h\mathbf{u}_\alpha)) - h\nabla(\nabla\cdot\mathbf{u}_\alpha)\right)\\ &+\frac{\mu^2}{2}\left[h^2B - \frac{(h+\delta\eta)^2}{3}\right]\nabla(\nabla\cdot\mathbf{u}_\alpha)\Bigg\}\end{aligned} \tag{2}$$

$$\begin{aligned}\mathbf{u}_{\alpha t} + \nabla\eta + \delta(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + \mu^2\left[(A-1)h\left(\nabla(\nabla\cdot(h\mathbf{u}_\alpha)) - h\nabla(\nabla\cdot\mathbf{u}_\alpha)\right)\right]_t&\\ +\frac{\mu^2}{2}\left[h^2(B-1)\nabla(\nabla\cdot\mathbf{u}_\alpha)\right]_t - \mu^2\nabla\left[\eta\nabla\cdot(h\mathbf{u}_{\alpha t}) + \frac{\eta^2}{2}\nabla\cdot\mathbf{u}_{\alpha t}\right]&\\ +\mu^2(\mathbf{u}_\alpha\cdot\nabla)\left[(A-1)h\left(\nabla(\nabla\cdot(h\mathbf{u}_\alpha)) - h\nabla(\nabla\cdot\mathbf{u}_\alpha)\right) + \frac{h^2}{2}(B-1)\nabla(\nabla\cdot\mathbf{u}_\alpha)\right]&\\ +\mu^2\left[(Ah-(\delta\eta+h))\left(\nabla(\nabla\cdot(h\mathbf{u}_\alpha)) - h\nabla(\nabla\cdot\mathbf{u}_\alpha)\right)\right.&\\ \left.+\frac{1}{2}(h^2B-(\delta\eta+h)^2)\nabla(\nabla\cdot\mathbf{u}_\alpha)\right]\cdot\nabla\mathbf{u}_\alpha&\\ -\mu^2\eta(\mathbf{u}_\alpha\cdot\nabla)\left[\nabla(\nabla\cdot(h\mathbf{u}_\alpha))\right] - \frac{\mu^2}{2}\eta^2(\mathbf{u}_\alpha\cdot\nabla)\nabla(\nabla\cdot\mathbf{u}_\alpha)&\\ +\frac{\mu^2}{2}\nabla\left[(\nabla\cdot(h\mathbf{u}_\alpha) + \eta\nabla\cdot\mathbf{u}_\alpha)^2\right]&\\ -\mu^2\nabla\eta\left[(\mathbf{u}_\alpha\cdot\nabla)(\nabla\cdot(h\mathbf{u}_\alpha)) + \eta(\mathbf{u}_\alpha\cdot\nabla)(\nabla\cdot\mathbf{u}_\alpha)\right] &= 0\end{aligned} \tag{3}$$

These equations also provide the basis for computations in Gobbi et al. (2000), which is concerned with the representation of vorticity in Boussinesq-type equations.

Improving linear properties

In the linearised equations, there are two degrees of freedom, ζ_1, ζ_2, that will allow us to impose Padé [2,2] dispersion while retaining the freedome to improve linear shoaling properties. Analysis of dispersive properties of the linearised equations on a flat bed yields the result $B = \frac{1}{5}$. Although in a different form, this result is identical to that of Nwogu (1993). To improve linear shoaling properties, a multiple scales expansion method was used for a mildly sloping bed (e.g. Madsen and Schäffer, 1998). However, the analysis was performed not only for shore-normal waves, but for varying deepwater incident wave angles. Results were plotted using the method of Chen and Liu (1995), using a range of shoaling parameter, and a value of $A = 0.425$ was chosen. Figure 3 shows results for the optimised value as well as results using the unmodified

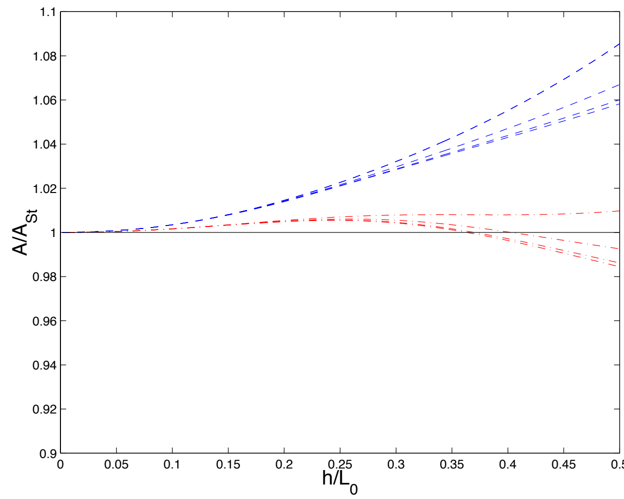

Figure 3: Linear shoaling performance for deepwater wave angles (starting from bottom on figure) $0, 15, 30, 45$ degrees. $(- \;-)$ WKGS; $(- \cdot -)$ WKGS + SH

WKGS equations. The improvement in shoaling properties is obvious. The set $(A = 0.425, B = 1/5$, denoted WKGS+SH) (or $\zeta_1 = -0.7142, \zeta_2 = -0.4358)$ will thus give Padé [2,2] linear dispersion as well as greatly improved linear shoaling.

Improving nonlinear properties

With the improvement to shoaling properties in the previous section, both linear shoaling and dispersion now give good results up to a a dimensionless wavenumber of $kh = 3$. However, nonlinear properties show significantly more error as depths increase. Fortunately, we are also able to manipulate independently nonlinear characteristics by choosing the portion of z_α that moves with the free surface in the form of β_1 and β_2. In common with linear characteristics, it should be possible to optimise nonlinear performance on both a flat and mildly sloping bed; however, only a flat bed was analysed due to the complexity of the nonlinear equations on a sloping bed. The resulting redundancy was removed by specifying $\beta = \beta_1 = \beta_2$, leaving only one degree of freedom.

This degree of freedom was set by comparing Boussinesq and exact Stokes-type expansions for steady waves. After equating asymptotically coefficients for the second bound harmonic to as great a degree as possible, the result $\gamma = 17/200$ is obtained, where $\gamma = \beta(1 + (\zeta_1 + \zeta_2)/2)$. Figure 4 shows a comparison between Boussinesq and exact second harmonics and setdown under a steady wave using the WKGS equations and the newly-optimised equations. The improvement in accuracy for for the bound second harmonic in intermediate depths is clear and, as will be seen later, also translates into additional accuracy for highly nonlinear waves in shallow water. However, equally clear is the lack of improvement in representing setdown under a steady wave. A more complete description of super-and-subharmonic properties may be found in Kennedy et al. (2000), in which only nonlinear optimisation is considered

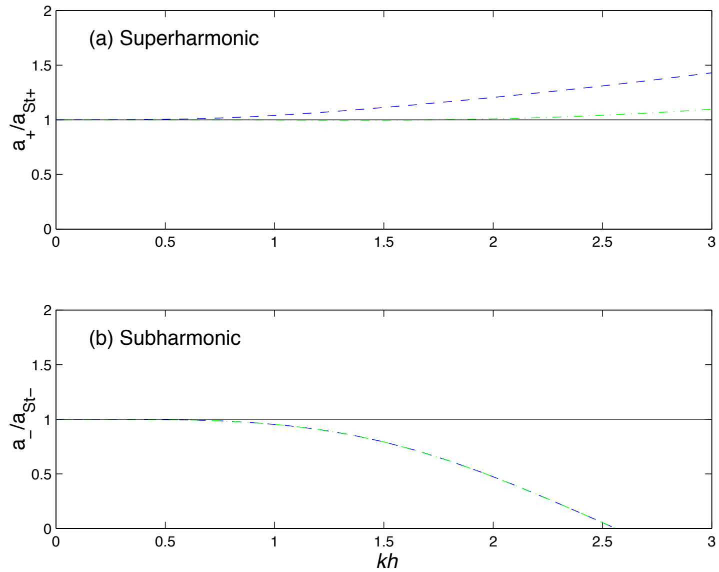

Figure 4: Self interaction superharmonics (a), and subharmonics (b) relative to full Stokes solution. $(--)$ WKGS; $(-\cdot-)$ WKGS+NL (WKGS+NL+SH identical)

(linear shoaling is not improved). The improvement in nonlinear properties is, to second order, independent of whether shoaling is improved and thus, a set with improved nonlinearity (WKGS+NL) may be defined in addition to the fully improved equations (WKGS+NL+SH).

Improving the MS equations: $O(\mu^4)$ equations only showing $O(\mu^2)$ terms

The MS equations (Madsen and Schäffer, 1998, Section 6) provide excellent Padé [4,4] dispersion on a flat bed and acceptable nonlinear properties. However, as derived, they are only $O(\mu^2)$ accurate, which would strictly donfine them to far shallower water. However, we will show that, unchanged, they are linearly of $O(\mu^4)$ accuracy for flat beds, and with small modifications, they can also provide $O(\mu^4)$ accuracy for wave transformation on mildly sloping beds.

To arrive at the improved MS equations, we will modify the equations of Gobbi and Kirby (1999). Linearly, for a flat bed, these are

$$\begin{aligned}\eta_t &+ h\nabla\cdot\tilde{\mathbf{u}} + \frac{\mu^2}{2}h^3\left(B-\frac{1}{3}\right)\nabla^2(\nabla\cdot\tilde{\mathbf{u}}) \\ &+ \frac{\mu^4}{4}h^5\left(B^2-\frac{B}{3}-\frac{D}{6}+\frac{1}{30}\right)\nabla^2\nabla^2(\nabla\cdot\tilde{\mathbf{u}}) = 0 \qquad (4)\\ \tilde{\mathbf{u}}_t &+ \nabla\eta + \frac{\mu^2}{2}h^2(B-1)\nabla(\nabla\cdot\tilde{\mathbf{u}}_t) \\ &+ \frac{\mu^4}{4}h^4\left(B^2-B-\frac{D}{6}+\frac{1}{6}\right)\nabla\nabla^2(\nabla\cdot\tilde{\mathbf{u}}_t) = 0 \qquad (5)\end{aligned}$$

The notation used by Gobbi and Kirby (1999) is almost identical to that of the previous section, but we will change it slightly and define

$$Bh^2 \equiv \frac{1}{4}\left[(h+z_{\alpha 1})^2+(h+z_{\alpha 2})^2+(h+z_{\alpha 3})^2+(h+z_{\alpha 4})^2\right] \tag{6}$$

$$Dh^4 \equiv \frac{1}{4}\left[(h+z_{\alpha 1})^4+(h+z_{\alpha 2})^4+(h+z_{\alpha 3})^4+(h+z_{\alpha 4})^4\right] \tag{7}$$

with A and C defined in a similar manner. For our purposes here, $z_{\alpha 1}$, etc., are all linear; i.e., $\beta_1 = 0$.

To the mass equation we now apply the operator $(1+\mu^2\psi_1\nabla\cdot(h^2\nabla(-)))$ and to the momentum equation $(1+\mu^2\rho_1 h^2\nabla(\nabla\cdot(-)))$ where ψ_1 and ρ_1 are constants to be determined. The resulting equations on a flat bed are

$$\begin{aligned}\eta_t &+ h\nabla\cdot\tilde{\mathbf{u}}+\frac{\mu^2}{2}h^3\left(B-\frac{1}{3}\right)\nabla^2(\nabla\cdot\tilde{\mathbf{u}})+\mu^2\psi_1 h^2\nabla^2(\eta_t+h\nabla\cdot\tilde{\mathbf{u}})\\ &+ \frac{\mu^4}{4}h^5\left(B^2-\frac{B}{3}-\frac{D}{6}+\frac{1}{30}+2\psi_1(B-\frac{1}{3})\right)\nabla^2\nabla^2(\nabla\cdot\tilde{\mathbf{u}})=0\end{aligned} \tag{8}$$

$$\begin{aligned}\tilde{\mathbf{u}}_t &+ \nabla\eta+\frac{\mu^2}{2}h^2(B-1)\nabla(\nabla\cdot\tilde{\mathbf{u}}_t)+\mu^2\rho_1 h^2\nabla(\nabla\cdot(\tilde{\mathbf{u}}_t+\nabla\eta))\\ &+ \frac{\mu^4}{4}h^4\left(B^2-B-\frac{D}{6}+\frac{1}{6}+2\rho_1(B-1)\right)\nabla\nabla^2(\nabla\cdot\tilde{\mathbf{u}}_t)=0\end{aligned} \tag{9}$$

We now set all $O(\mu^4)$ terms to zero by specifying

$$\psi_1=-\frac{B^2-\frac{B}{3}-\frac{D}{6}+\frac{1}{30}}{2(B-\frac{1}{3})}, \qquad \rho_1=-\frac{B^2-B-\frac{D}{6}+\frac{1}{6}}{2(B-1)} \tag{10}$$

The resulting family of equations will show no $O(\mu^4)$ terms in either the mass or momentum equation, even though the system is linearly of $O(\mu^4)$ on a flat bed. The two remaining free parameters, B and D, may thus be chosen to retain Padé [4,4] dispersion. After some calculation, it is seen that this dispersion may be achieved by four sets of parameters, which exactly correspond to sets I-IV given by MS. Here, set I will be used exclusively ($B = 0.21048, D = 0.1026, \psi_1 = -0.039166, \rho_1 = -0.010520$), as it is the only one with acceptable nonlinear properties. Thus, even though the MS equations were derived as $O(\mu^2)$ equations, they are found to be linearly of $O(\mu^4)$ on a level bed. The improved equations will have identical properties to MS on a flat bed, and thus Madsen and Schäffer (1998) may be consulted for a more detailed description of constant-depth properties.

A similar, though much more complex, process is used to cancel linear $O(\mu^4)$ terms for slopes of order ∇h. Because of this complexity and length limitations, only the final results will be shown. For a more complete derivation, consult Kennedy et al. (2000c).

In contrast to the situation on a flat bed, where no assumption was made about vertical vorticity, on a mildly sloping bed we will explicitly assume irrotational flow.

To cancel $O(\mu^4)$ terms in the mass equation, it would be possible to apply an operator identical to that of MS and then choose an additional free parameter in order to cancel higher order terms. In fact, there are an entire family of operators that could be used. Here, we have chosen one which simplifies the analysis somewhat when compared to that found in MS. The operator applied to the mass equation is

$$\left(1 + \mu^2\psi_1\nabla\cdot\left[h^2\nabla(-)\right] + \mu^2\psi_2\nabla\cdot\left[h\nabla h(-)\right]\right) \tag{11}$$

which, on a flat bed reduces to that given previously, and thus (10) still applies. On a mildly sloping bed, imposing

$$\psi_2 = \frac{-\left[B(A-\frac{1}{2})+(B-\frac{1}{3})(A+\frac{9}{4}B)-\frac{2}{3}(C-\frac{1}{4})-\frac{5}{24}(D-\frac{1}{5})\right]-\psi_1\left[2(A-\frac{1}{2})+\frac{11}{2}(B-\frac{1}{3})\right]}{\frac{1}{2}(B-\frac{1}{3})} \tag{12}$$

will analytically cancel all $O(\mu^4)$ linear terms in the mass equation up to order ∇h, thus achieving our goal.

For the conservation of momentum equation, the situation is somewhat more complicated. For the mass equation, there was only one type of term that was found for mild slopes at $O(\mu^4)$, and thus only one additional term needed for the differential operator. For the conservation of momentum equation, there are two additional types of terms found for mild slopes and thus two additional terms are required in the differential operator. The operator found in MS has only one additional term to account for mild slopes and thus is incapable of cancelling analytically all $O(\mu^4)$ terms in two dimensions. (The MS operator is, however, capable of cancelling all linear mild slope terms in one dimension.) The operator applied to the momentum equation here is

$$1 + \mu^2\rho_1 h^2\nabla(\nabla\cdot(-)) + \mu^2\rho_2 h\nabla h(\nabla\cdot(-)) + \mu^2\rho_3\nabla(h\nabla h\cdot(-)) \tag{13}$$

where ρ_1 is specified by (10) and

$$\rho_2 = \frac{-\rho_1(B-1) - \left[(A-1)\frac{B}{2} + (B-1)\frac{B}{2} - \frac{1}{6}(C-1)\right]}{\frac{1}{2}(B-1)} \tag{14}$$

$$\rho_3 = \frac{-\rho_1\left[2(A-1)+(B-1)\right] - \left[(A-1)\frac{B}{2} + (B-1)(A+\frac{B}{2}) - \frac{1}{2}(C-1)\right]}{\frac{1}{2}(B-1)} \tag{15}$$

The free coefficients A and C may now be chosen in order to best improve linear shoaling.

Once again, the free shoaling coefficients were chosen based on the results of a multiple scales expansion similar to that found in Madsen and Schäffer (1998). Because of algebraic complexity, the details of the analysis will not be shown. Figure 5 shows shoaling properties for shore-normal waves using coefficients $A = 0.404$, and $C = 0.1431$ as well as the preferred results from MS. The improvement is clear.

The coefficients used are thus $A = 0.404, B = 0.21048, C = 0.1431, D = 0.1026, \psi_1 = -0.039166, \psi_2 = -0.040293, \rho_1 = -0.010520, \rho_2 = 0.013448, \rho_3 =$

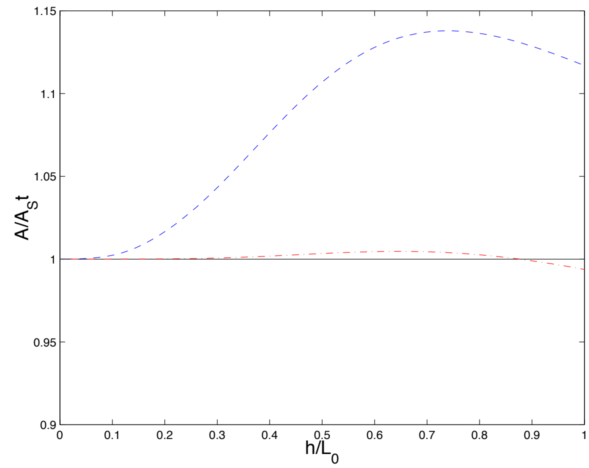

Figure 5: Linear shoaling performance for shore-normal waves, $(-\ -)$ MS; $(-\cdot-)$ Improved MS

-0.039222. For practical use, it is simpler to treat the new equations as extensions to the improved $O(\mu^2)$ equations introduced earlier in the paper, since these equations include a more complete representation of vertical vorticity. Thus the operators (11,13) may be applied to (1-3) and the resulting $O(\mu^4)$ terms ignored.

Numerical tests of nonlinear wave shoaling

To test the theoretical improvements, computational experiments were performed to test nonlinear wave shoaling. Boussinesq results using the various formulations were compared with results using the potential flow method of Kennedy and Fenton (1997), which are essentially numerically exact. The tests considered were initial value problems similar to those in Kennedy et al. (2000b). In this, a specified initial surface elevation with zero initial fluid velocity, was allowed to evolve over a given topography for a set time. Figure 6 shows an example of the initial and final surface elevations, and the underlying topography. Also shown are the crest and trough envelopes, the highest and lowest elevations reached during the simulation. Boussinesq results for these envelopes were compared with potential flow values.

The topography was given by

$$\begin{aligned} \frac{h}{h_1} &= \frac{h_{min}}{h_1} + (1 - \frac{h_{min}}{h_1}) \left[1/\cosh\left(\tan(\frac{\pi x}{2L})\right)\right], \quad x \leq L \\ \frac{h}{h_1} &= \frac{h_{min}}{h_1}, \quad L < x \leq 2L \end{aligned} \tag{16}$$

where $h_{min}/h_1 = 0.2$, and $L = 50h_1$. Vertical walls were specified at the left and right boundaries. The initial surface elevation was

$$\frac{\eta}{h_1} = \frac{a_I}{h_1} \left[\frac{\cos(\frac{2\pi N_w x}{L})}{\cosh\left(\tan(\frac{\pi x}{2L})\right)} \right], \quad x \leq L$$

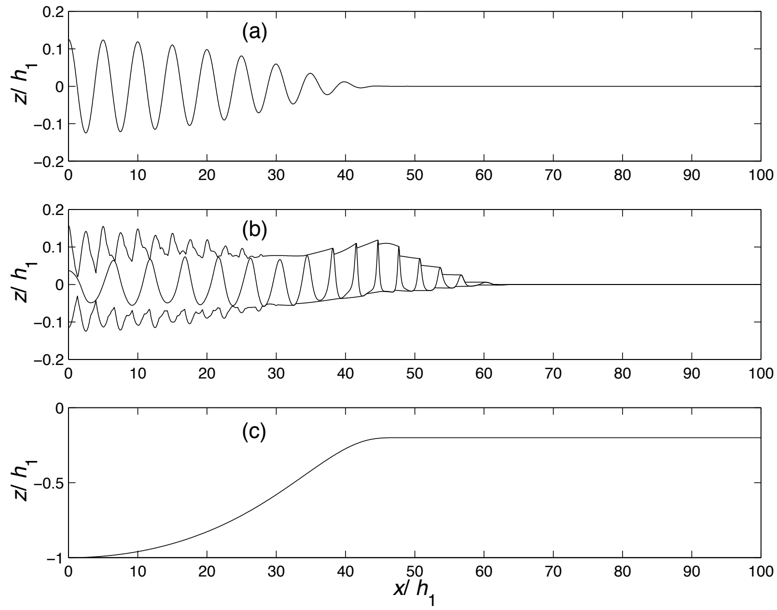

Figure 6: (a) Initial and (b) final surface elevations for case A, with crest and trough envelopes; (c) underlying bathymetry

$$\frac{\eta}{h_1} = 0, \quad L < x \leq 2L \tag{17}$$

Figure 7 shows results for case A, with initial wave amplitude $a_I/h_1 = 0.125$, initial number of waves $N_w = 10$, and simulation time $T\sqrt{g/h} = 45$. This case was more a test of highly nonlinear wave evolution in shallow water, which was where the original WKGS equations showed error (see Figure 2), than of linear shoaling. Envelopes are shown only from $40 < x/h_1 < 60$ as it is in this area that wave height to depth ratios are largest. The first point of interest is that trough envelopes are nearly identical, as wave nonlinearity is concentrated in the crest. This concentration of nonlinearity in the wave crest is ubiquitous for shallow water, near-breaking waves.

The most noticeable feature in the simulations was the difference in crest envelopes as the waves become very large. The WKGS equations tend to overpredict significantly crest elevation when compared to the numerically exact potential flow solution. This is identical to the behaviour shown earlier. However, the $O(\mu^2)$ equations with improved shoaling and nonlinearity show much better behaviour and only show small differences from the potential flow solution. The equations only using improved nonlinearity are almost identical, but this is expected since shoaling began in quite shallow water. It thus seems that the asymptotic improvement in mildly nonlinear properties also improves highly nonlinear properties in shallow depths, which is a welcome finding. The MS equations and improved MS equations also show good, and nearly identical, behaviour, although not as good as the improved WKGS equations in this case.

The first case tested nonlinear wave behaviour near breaking, and showed clearly

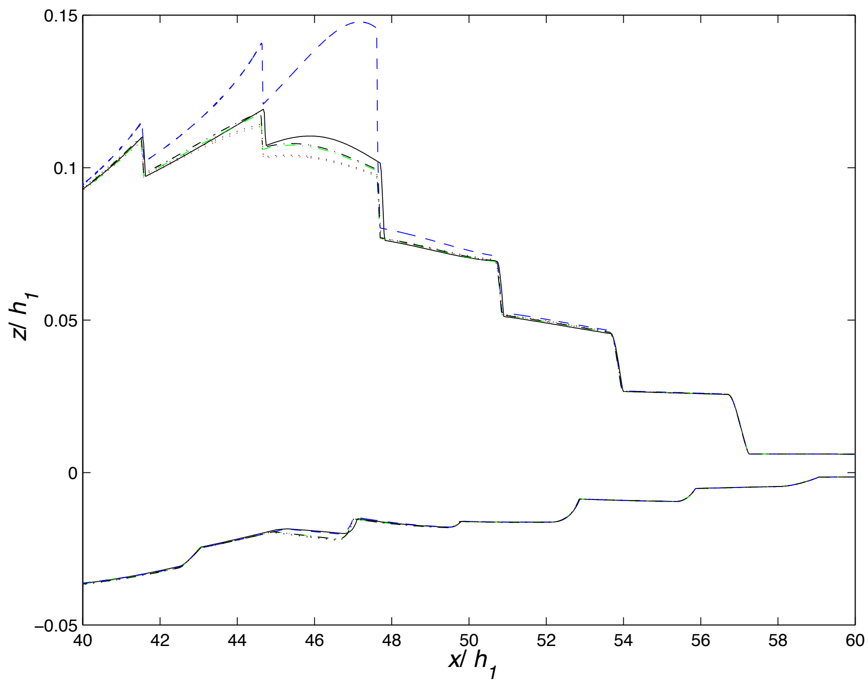

Figure 7: Crest and trough envelopes for unsteady shoaling, case A. $(-)$ accurate potential flow; $(--)$ WKGS; $(\cdots)$ MS and improved MS (almost identical); $(-\cdot-)$ WKGS+NL and WKGS+NL+SH (almost identical)

the great improvement in practical performance brought about by the more accurate representation of higher harmonics in the improved WKGS equations. Now we will consider case B, where shoaling is less nonlinear, and thus shoaling properties (from starting wavenumber $k_1h_1 \approx 2.5$) are more important than nonlinearity. This may be represented as $a_I/h_1 = 0.05$, $N_w = 20$, and $T\sqrt{g/h} = 100$.

Figure 8 shows results for all sets of equations. Wave heights are plotted somewhat differently here with dimensionless distance plotted against the envelope of wave height. This was done as in this case, trough envelopes as well as crest envelopes differed from the exact solution.

Here, for the first time, we see differences between the nonlinearity-improved WKGS equations and the shoaling-and-nonlinearity-improved WKGS equations. The equations with improved shoaling show significantly less error than the equations without improved shoaling, although both results are reasonable. The original WKGS equations here show results almost indistinguishable from the nonlinearity-improved WKGS equations. As this is a test of unsteady shoaling, it is likely that errors in both sets of results are beginning to be affected by errors in the linear dispersion relationship. The MS equations, surprisingly, show significant error using the recommended shoaling coefficients. However, the improved MS equations, show excellent behaviour, with shoaling wave heights that are almost indistinguishable from the exact solution for

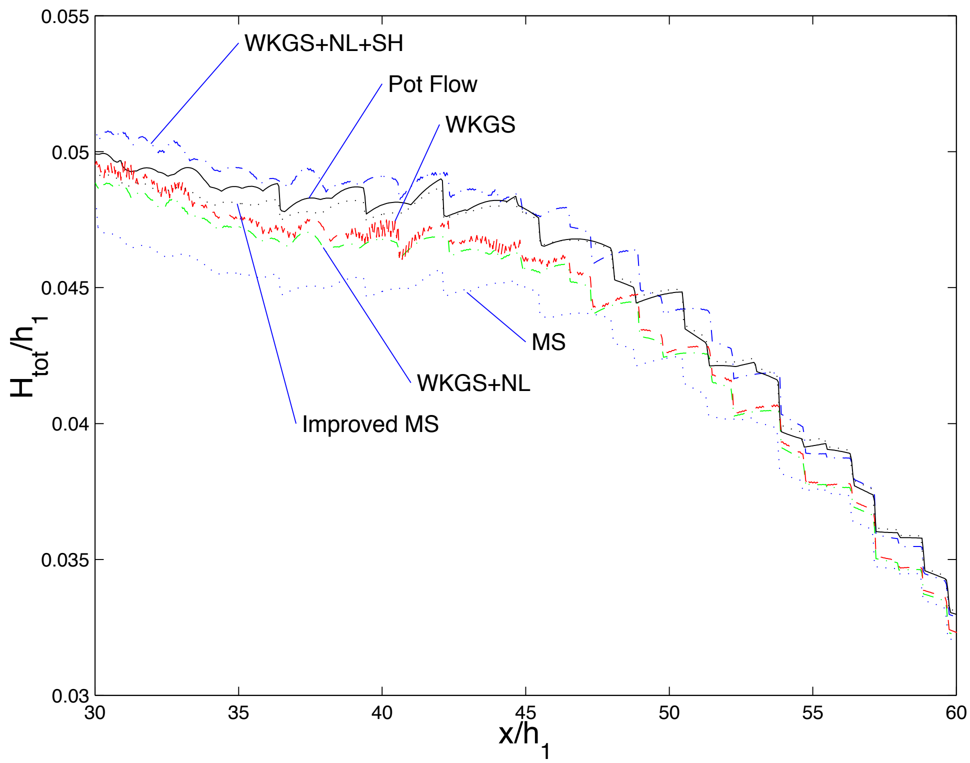

Figure 8: Wave height envelope for unsteady shoaling, case B.

much of the domain. This vastly improved performance is partly due to the increased accuracy resulting from the new formulation, but is also partly due to the method MS used to optimise the free shoaling coefficients. Their technique was found by Chen and Liu (1995) to overemphasise shoaling errors in deep water at the expense of shallow water accuracy. As a final note, in this test, the advantage of Padé [4,4] dispersion becomes evident, as the MS and improved MS equations show an almost exact phase correspondence with the detailed peaks in the wave height envelope, while the less accurate Padé [2,2] dispersion in the WKGS and improved WKGS equations gives a far less detailed picture.

Discussion and Conclusions

To some degree, the choice of improved WKGS equations or improved MS equations is a matter of preference. The improved WKGS equations have excellent nonlinear properties up to the breaking limit and are somewhat simpler when compared to the improved MS equations. However, the improved MS equations have quite reasonable nonlinear performance, and have far superior dispersive and shoaling properties from deep water. If phase accuracy is important, the Padé [4,4] accuracy in the improved MS equations will also become a factor. However, either set of equations will give good results for wave transformation up to the nominal deep water limit of $kh = \pi$.

With results here and elsewhere, it is becoming very clear that the Boussinesq approach can now be used to model with quite good accuracy both linear and nonlinear wave transformation. It also seems likely, although the mechanism is somewhat obscure, that improving mildly nonlinear superharmonic properties of fully-nonlinear Boussinesq equations in intermediate depths will also improve highly nonlinear properties in shallow depths. No technique for improving subharmonic accuracy was introduced here, although very recent work has revealed a technique which offers the capability to improve both superharmonic and subharmonic accuracy (Kennedy et al., 2000c).

It also seems apparent that achieving $O(\mu^4)$ accuracy while only showing $O(\mu^2)$ terms is a worthwhile goal that can result in increased accuracy for Boussinesq-type equations. Although only linear $O(\mu^4)$ terms were considered here, it seems likely that some form of nonlinear $O(\mu^4)$ behaviour may also be represented using related techniques. This work is in progress and will be reported on in the future.

In conclusion, both the improved WKGS equations and improved MS equations introduced here show properties which make them more desirable than their original forms. As an added bonus, both sets of equations are very similar to their original forms, with only small changes producing these improved properties. Computationally, these changes are somewhat between small and trivial, depending on the details of the scheme used. However, the rewards are undeniable.

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