# CHAPTER 10

# Waves in an Annular Entrance Channel

Robert A. Dalrymple, F. ASCE and James T. Kirby, M. ASCE[1]

**Abstract**

Waves propagating in a curved channel are examined analytically and with a variety of parabolic and spectral models. The results show that the wide angle parabolic method is reasonably robust, but not exact, while a spectral method based on trigonometric functions is superior to a Chebyshev polynomial method. Angular spectrum models are discussed and shown to be equivalent to an eigenfunction expansion in the cross-channel direction.

## Introduction

Waves propagating in curved narrow (with respect to the wave length) channels behave as if the channel were straight. However, when the curved channel is wide, then wave reflection from the outer wall and the diffraction of waves around the channel bend at the inner wall lead to a complicated sea state within the channel.

The prediction of the transmission of waves into harbors with curved entrance channels and the use of annular wave tanks (in small laboratories) depend on our ability to model waves in curved domains. Here waves propagating in a curved channel, taken to be a section of a circular annulus, are studied. An analytical and three different types of numerical solutions are obtained and compared. The analytical solution is used to verify the validity of the numerical techniques developed in a conformally mapped domain, resulting from mapping the curved channel into a straight channel.

An annular channel can be classified as narrow or wide with respect to the incident wave depending on the dimensionless width, defined by Dalrymple, Kirby, and Martin (1995) as $k(r_2 - r_1) = kw$, where $r_2$ is the outer radius of the channel, $r_1$ is the inner radius, $w$ is the width of the channel, and $k$ is the wavenumber of the progressive wave train found from the dispersion equation,

$$\sigma^2 = gk \tanh kh$$

[1]Center for Applied Coastal Research, Ocean Engineering Laboratory, Univ. Delaware, Newark, DE 19716





where the angular frequency of the wave is $\sigma = 2\pi/T$, the wavenumber $k = 2\pi/L$, $h$ is the water depth, and $g$ the acceleration of gravity. The parameter $kw$ is $2\pi$ times the number of waves that fit across the channel width; wide channels can fit numerous waves across their widths.

In acoustics, the problem of sound propagation in curved ducts has been studied by Rostafinski in a variety of papers (see References). The mathematical problem is analogous, although now the computing tools make the problem much simpler.

## Analytical Solution

For linear wave theory, with irrotational motion and an incompressible fluid, the governing equation for the velocity potential of the wave motion is the Laplace equation. The Laplace equation is written conveniently for this problem in polar form; therefore, we will take the velocities to be given by

$$u = -\frac{\partial \Phi}{\partial r} \qquad (1)$$

$$v = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} \qquad (2)$$

$$w = -\frac{\partial \Phi}{\partial z} \qquad (3)$$

and the Laplace equation is:

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad (4)$$

Figure 1 shows a layout of an annular channel with a maximum $\theta$ value of $\pi$, corresponding to the positive $y$ axis.

Taking the depth $h$ as constant, we assume $\Phi(r,\theta,z) = \phi(r,\theta)\cosh k(h+z)$, where the wavenumber $k$ is fixed from the dispersion relationship. Substituting into (4), gives the polar form of the Helmholtz equation:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + k^2\phi = 0 \qquad (5)$$

The boundary conditions for vertical no-flow channel sidewalls are

$$\frac{\partial \phi}{\partial r} = 0, \text{ at } r = r_1, \text{ the inner wall} \qquad (6)$$

$$\frac{\partial \phi}{\partial r} = 0, \text{ at } r = r_2, \text{ the outer wall.} \qquad (7)$$

The general solution for $\phi$ that satisfies these boundary conditions is

$$\phi = \sum_{n=0}^{N} a_n \left[ Y'_{\gamma_n}(kr_1) J_{\gamma_n}(kr) - J'_{\gamma_n}(kr_1) Y_{\gamma_n}(kr) \right] e^{i\gamma_n \theta} \qquad (8)$$

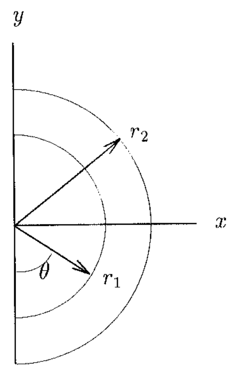


Figure 1: Schematic Diagram of Circular Channel

$$= \sum_{n=0}^{N} a_n F_n(r)\, e^{i\gamma_n \theta} \qquad (9)$$

where $k$ is the wavenumber of the incident wave, $r_1$ is the inner radius of the channel, and $\gamma_n$ must satisfy

$$Y'_{\gamma_n}(kr_1)J'_{\gamma_n}(kr_2) - J'_{\gamma_n}(kr_1)Y'_{\gamma_n}(kr_2) = 0, \ \text{n=1, 2,..., N}$$

to enforce a no-flow boundary condition on the outer wall ($r = r_2$)(see, e.g. Kirby *et al.*, 1994). There are only $N$ real values of $\gamma_n$ that satisfy this equation, which are ranked in descending order of magnitude. The largest $\gamma_n$ is less than $kr_2$.

Each of the terms in the summation (Eq. 9) is a wave mode, propagating in the $\theta$ direction with $1/\gamma_n$ waves per $2\pi$ radians, and a cross-channel variation given by the radial term $F_n(r)$. These terms are orthogonal to each other with weight $(1/r)$ from the Sturm-Liouville theorem and provide a method for determining the $a_n$ values.

The values of the unknown coefficients $a_n$ in Eq. 9 are found from the initial condition at $\theta = 0$, which is prescribed as a function of $r$. Here we assume that the wave height is uniform across the mouth of the channel. In actuality, there is an interaction between the channel and the ocean that leads to variations across the channel. These are assumed to be small. For $\phi(r,0) = 1$, we use the expression for the velocity potential (9) and the orthogonality of the $r$ terms to find the $a_n$.

$$a_n = \frac{\int_{r_1}^{r_2} r^{-1}\phi(r,0)F_n(r)dr}{\int_{r_1}^{r_2} r^{-1}F_n^2(r)dr} \qquad (10)$$



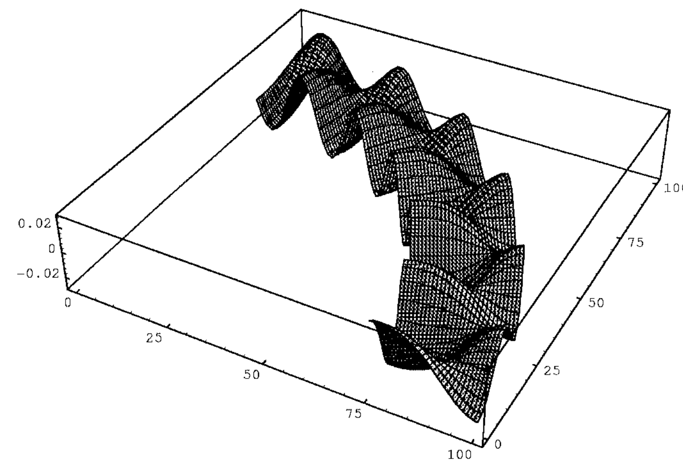


Figure 3: First Mode of Analytical Solution

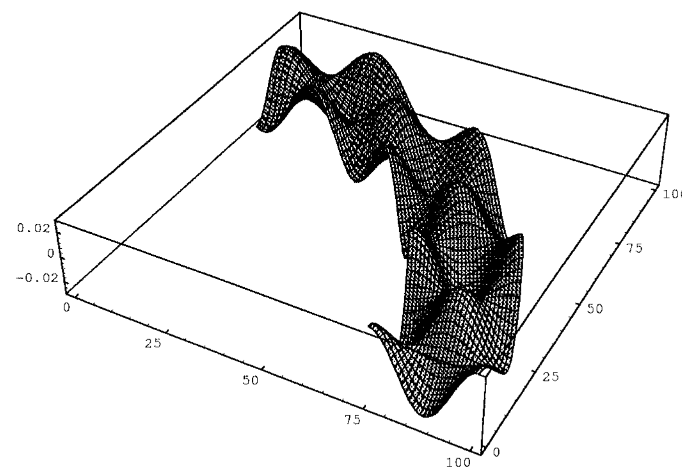


Figure 4: Second and Last Mode for Analytical Solution

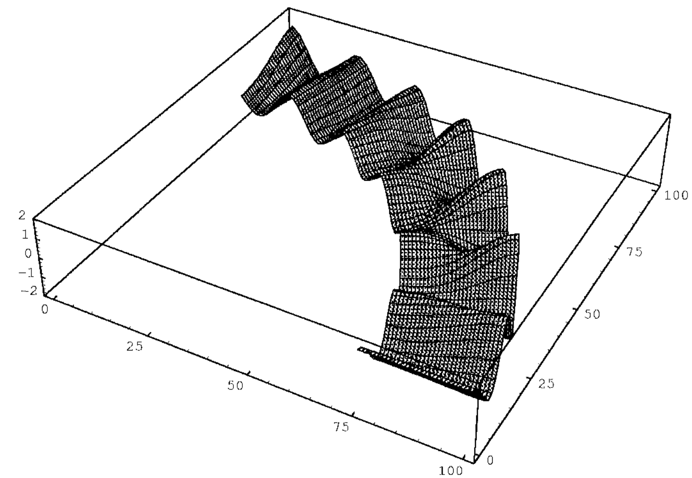


Figure 5: Analytic Solution (Sum of All Three Modes)

seiching problem was studied by Campbell (1953), who had to deal with waves in a circular (ship) testing channel.

## Parabolic Modelling

A simple parabolic model can be easily derived from the Helmholtz equation (5). Since the waves will travel in the azimuthal direction (particularly for narrow channels), most of the oscillation in the wave form can be described by a periodic function in $\theta$ and we write

$$\phi(r,\theta) = A(r,\theta)\, e^{i\Gamma\theta} \tag{11}$$

where $\Gamma$ is a dimensionless constant and $A$ is likely to vary slowly in the $\theta$ direction. Substituting into the Helmholtz equation, we find, after neglecting a second derivative of $A$ with respect to $\theta$ (assumed to be small).

$$2ikr_0\frac{\partial A}{\partial \theta} + (k^2r^2 - k^2r_0^2)A + r\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) = 0 \tag{12}$$

The value of $\Gamma$ was taken as $kr_0$, where $r_0$ is a reference radius arbitrarily taken as the mid-point of the channel, $\overline{r}$, or $r_0 = \overline{r} = (r_1 + r_2)/2$. This model will work for narrow channels.



We can obtain more correct values of $r_0$ by comparing to the $\gamma_1$ angular wavenumbers from the analytical solution. By comparing to a number of cases, we find an approximate linear trend (with unit slope) between $(r_0 - \overline{r})/\overline{r}$ with the relative width of the channel $k(r_2 - r_1)$. Therefore for narrow channels, $r_0 = \overline{r}$, but for wide channels, $r_0 = \overline{r}(1 + k(r_2 - r_1))$.

Kirby, Dalrymple, and Kaku (1994) provide a general form for small and large angle parabolic models in arbitrary mapped and conformal domains. Here we need only the mapping for conformal domains.

The constant depth Helmholtz equation with constant coefficients is conformally mapped into a variable coefficient Helmholtz equation:

$$\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2} + k^2J(u,v)\phi = 0 \tag{13}$$

Here, $J(u,v)$ is the Jacobian of the transformation. Assuming that $\phi(u,v)$ varies rapidly in the propagation direction $u$, we write

$$\phi(u,v) = Re\left\{A(u,v)e^{i\int k_0 J_0^{1/2} du}\right\} \tag{14}$$

where $k_0 = k(u,v_0)$ and $J_0 = J(u,v_0)$ and $v_0$ is a fixed reference distance. Substituting into Eq. 13 and neglecting a second derivative of $A$ with respect to $u$, Kirby *et al.* find a small angle parabolic model:

$$2ik_0J_0^{1/2}\frac{\partial A}{\partial u} + i\frac{\partial(K_0J_0^{1/2})}{\partial u}A + (K^2J - K_0^2J_0)A + \frac{\partial^2 A}{\partial v^2} = 0 \tag{15}$$

In wide channels, the turning of the channel leads to large angle discrepancies between the assumed azimuthal propagation direction and the actual wave direction. To allow for these wave directions differing from the azimuthal direction to a greater extent than permitted by the small angle parabolic model, a wide angle model was developed:

$$2ikJ^{1/2}\frac{\partial A}{\partial u} + 2kJ^{1/2}\left(kJ^{1/2} - k_0J_0^{1/2}\right)A + i\frac{\partial(kJ^{1/2})}{\partial u}A + \left\{\frac{3}{2} - \frac{1}{2}\left(\frac{k_0^2J_0}{k^2J}\right)^{1/2}\right\}\frac{\partial^2 A}{\partial v^2} - \frac{3i}{4k^2J}\frac{\partial(kJ^{1/2})}{\partial u}\frac{\partial^2 A}{\partial v^2} + \frac{i}{2kJ^{1/2}}\frac{\partial^3 A}{\partial uvv} = 0 \tag{16}$$

Kirby *et al.* use a conformal mapping to convert a circular channel into a straight channel in the $(u,v)$ domain. One such map is $w = u + iv = \pi/2 - i\ln(z/r_m)$, where $r_m = \sqrt{r_1r_2}$. For this case, the $kJ_0^{1/2}$ is not a function of $u$ and the equations simplify. For the small angle parabolic models, it is easily shown that Eq. 12 and 15 are equivalent. They compared their models to the exact solution for a wide angle case; the simple parabolic began to fail 60°from the channel mouth, while the wide angle model compared reasonably well for the full 180°test channel. They also went further, adding nonlinear effects to the models.

## Spectral Modelling

### Fourier-Galerkin Modelling

Angular spectrum modelling has been used to model waves propagating over sloping and irregular bathymetry in regions which are bounded by a rectangular box (cf. the review by Dalrymple and Kirby, 1992). The angular spectrum model was defined by Booker and Clemmow (1950) in a Cartesian coordinate system as the decomposition of the initial condition into progressive plane waves. The subsequent wave field is found by summing the plane waves within the computational domain. In an $(x, y)$ coordinate system, the angular spectrum corresponds to a Fourier-Galerkin spectral method of solution, as the plane waves in this system are described by trigonometric functions. In the polar coordinate system, the angular spectrum model corresponds to a modal decomposition, based on the analytical modes shown above. Therefore the angular spectrum model in any coordinate system corresponds to an eigenfunction expansion for that coordinate system.

The Fourier-Galerkin model (using trigonometric functions in the cross-channel (radial) direction) is a natural extension of the angular spectrum model in Cartesian coordinates (e.g., Dalrymple *et al.*, 1989): it is based on the Fourier transform of the wave field in the lateral direction. In the Fourier domain, the wave equations are split into forward and backward propagating waves. Only the forward waves are kept as back-reflection is assumed to be negligible.

Dalrymple, Kirby and Martin (1995) have used Fourier-Galerkin modelling to examine waves in conformally mapped channels. In the $(u, v)$ conformal domain, the Fourier transform pair is:

$$f_n(u) = \mathcal{T}_F[\phi(u,v)] = \frac{1}{2v_b}\int_{-v_b}^{v_b} \phi(u,v)\cos[\lambda_n(v+v_b)]\,dv, \quad n = 0,1,2,. \tag{17}$$

$$\phi(u,v) = \mathcal{T}_F^{-1}[f(u)] = \sum_{n=0}^{\infty} \epsilon_n f_n(u)\cos[\lambda_n(v+v_b)] \quad \text{for } -v_b < v < v_b. \tag{18}$$

where $v_b = \frac{1}{2}\ln(r_2/r_1)$, which is half the channel width in the conformal domain.

Defining

$$\overline{k^2J}(u) = \frac{1}{2v_b}\int_{-v_b}^{v_b} k^2 J(u,v)\,dv. \tag{19}$$

and substituting this into the governing Helmholtz equation gives

$$\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2} + \overline{k^2J}\,(1-\nu)\phi = 0, \tag{20}$$

where

$$\nu(u,v) = 1 - k^2J/\overline{k^2J} \tag{21}$$



Fourier transforming the above Helmholtz equation yields

$$\frac{d^2 f_n}{du^2} + \gamma_n^2 f_n - \overline{k^2J}\,\mathcal{T}_F[\nu\,\mathcal{T}_F^{-1}[f]] = 0, \qquad n = 0,1,2,\ldots, \tag{22}$$

where

$$\gamma_n^2(u) = \overline{k^2J} - \lambda_n^2 \quad \text{and} \quad \lambda_n = \tfrac{1}{2}n\pi/v_b.$$

Splitting the problem into forward and backward propagating modes, they find the following equation for the forward propagating modal amplitudes, $f_n$, in the Fourier-transformed conformal domain:

$$\frac{df_n(u)}{du} = \mathrm{i}\gamma_n f_n - \frac{\mathrm{i}\,\overline{k^2J}}{2\gamma_n}\mathcal{T}_F[\nu\,\mathcal{T}_F^{-1}[f]], \qquad n = 0,1,2,\ldots \tag{23}$$

where

$$\overline{k^2J} = \frac{k^2(r_2^2 - r_1^2)}{2\ln(r_2/r_1)} \quad \text{and} \quad \nu(v) = 1 - \frac{2\ln(r_2/r_1)\,r_1 r_2 e^{-2v}}{r_2^2 - r_1^2},$$

This set of first order ordinary differential equations is solved numerically.

After the conformal mapping, the cosines used in the Fourier transform are no longer an optimal basis, as the forms of the lateral eigenfunctions $F_n(r)$ are far more complicated, as shown in Eq. 9. The errors increase with the width of the channel. This has implications also for application of the angular spectrum model for open coast cases where the bathymetry is varying significantly in the longshore direction.

### Chebyshev-tau Modelling

Chebyshev polynomials provide another set of orthogonal bases with which to expand the wave potential across the channel. Panchang and Kopriva (1989) have used these polynomials for the solution of the mild-slope equation. Here a cross-channel Chebyshev transform is used to develop the governing equation in the transform domain, which must be scaled to be in the range of -1 to 1, so we define $\zeta = v/v_b$, so that the lateral boundaries are located at $\zeta = \pm 1$. The appropriate Chebyshev-transform pair is:

$$c_n(u) = \mathcal{T}_C[\phi(u,v)] = \frac{\epsilon_n}{\pi}\int_{-1}^{1} \frac{\phi(u, v_b\zeta)\,T_n(\zeta)}{\sqrt{1-\zeta^2}}\,d\zeta, \quad n = 0,1,2,\ldots, \tag{24}$$

$$\phi(u,v) = \mathcal{T}_C^{-1}[c(u)] = \sum_{n=0}^{\infty} c_n(u)\,T_n(\zeta) \qquad \text{for } -1 < \zeta < 1. \tag{25}$$

As the Chebyshev polynomials do not satisfy the lateral boundary conditions, the tau method, which forces the Chebyshev sum to satisfy these conditions, Is used.

Introducing $\overline{k^2J}$ and $\nu$, defined by (19) and (21), respectively, we find that the Chebyshev transform of (20) is

$$\frac{d^2 c_n(u)}{du^2} + \overline{k^2 J}\, c_n(u) + \frac{1}{v_b^2} c_n^{(2)} - \overline{k^2 J}\,\mathcal{T}_C[\nu\, \mathcal{T}_C^{-1}[c]] = 0.$$

The splitting in the transform domain yields:

$$\frac{dc_n^+}{du} = \mathrm{i}\gamma_0 c_n^+(u) + \frac{\mathrm{i}}{2\gamma_0}\left(\frac{1}{v_b^2} c_n^{(2)} - \overline{k^2 J}\,\mathcal{T}_C[\nu\, \mathcal{T}_C^{-1}[c^+]]\right) = 0. \qquad (26)$$

There are significant disadvantages of the Chebyshev approach. First all modes are progressive, while for the Fourier-Galerkin method, only those modes for which $\gamma_n$ are positive propagate (thus reducing the number of simultaneous differential equations to solve). Also, the splitting of the Chebyshev transformed equation introduces an error, which does not occur with the Fourier-Galerkin model; it in fact reduces the Chebyshev method to an equivalent of the small angle parabolic model (Dalrymple *et al.*, 1995).

## Results

For narrow channels, all of the methods, parabolic and spectral, work well. As the channel width $kw$ increases, then the errors begin increase for all the methods. As an example of these errors, we increase the width in the previous example to $w$= 125 m ($r_1$= 75 m, $r_2$ = 200 m); now $kw$ = 37.625, or almost 6 wavelengths can fit across the channel.

The exact solution, given by the analytical model, is shown in Figure 6. The waves entering the channel from the right, begin to diffract around the channel bend and to reflect from the outer radius. At about 120°, the wave field is dominated by reflected waves from the outer side wall. Finally at 180°, the wave field is reasonable complex, indicating that a 180 °bend in a laboratory wave channel will not result in planar waves.

The surface elevations along the outer wall ($r = r_2$) predicted by the exact solution and the small and wide angle parabolic models are shown in Figure 7. The wide angle parabolic model does a reasonable job for the 180°length of channel shown, while the small angle begins to fail at 50°. Note that the phase of the waves is well predicted by the parabolic model.

For the Fourier-Galerkin model, a comparison to the exact solution for water surface elevations at the outer radius is shown in Figure 8 for 90°, while the Chebyshev result is shown in Figure 9. Both have problems with the phase speed of the wave as evidenced by the mis-matching of the wave crests. The Fourier-Galerkin results have the biggest discrepancy about 52°, but the Chebyshev model fairs worse, with mismatches over the entire sector from 50-90 °.



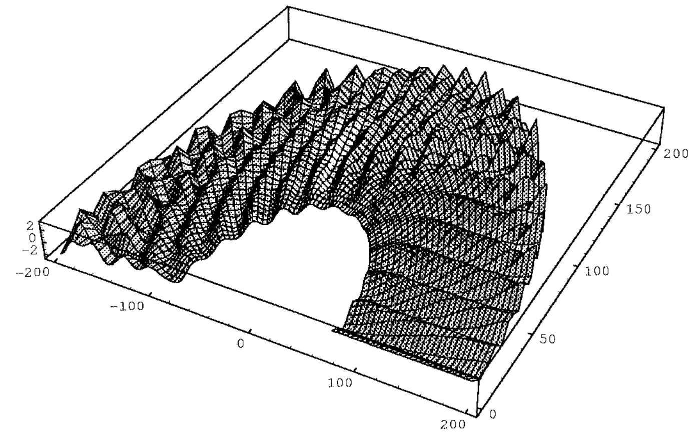


Figure 6: Exact Solution for Waves in a Wide Circular Channel

## Conclusions

Waves propagating in curved channels are shown for constant depth channels that are annular in planform. For a given wave height at the mouth of the channel, the wave field within the channel is predicted by an analytic solution and parabolic and spectral (Fourier-Galerkin and Chebyshev-tau) methods.

The analytic solution, by separation of variables, shows the modal structure of the wave field. The lowest mode, with no zero crossings across the channel, combines with higher order modes to form the total wave field. The wider the channel the more modes are present with the largest mode having an angular wavenumber $\gamma_1$ less than $kr_2$. Higher modes have smaller values of angular wavenumber.

All of the numerical models work well for narrow channels (say $kw < 8$) and become more inaccurate as the dimensionless width of the channel increases, with the wide-angle parabolic model doing better than the Fourier-Galerkin model, which out-performed the Chebyshev-tau model for the wide channel case ($kw = 37.6$).

## Acknowledgments

Both authors were supported in part by the NOAA Office of Sea Grant,

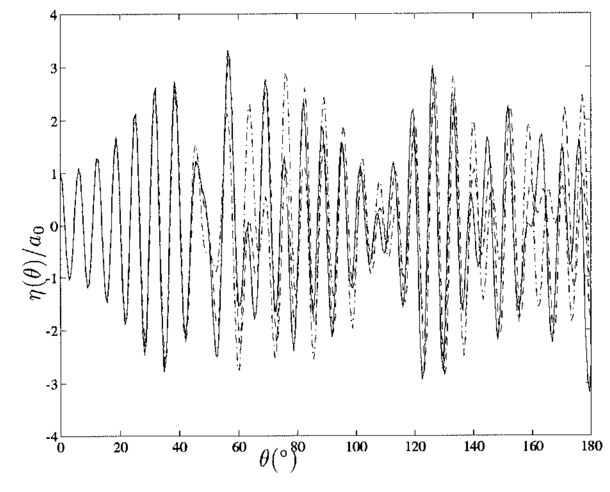


Figure 7: Parabolic model Results. Analytical solution is the solid line; dashed line, wide angle parabolic model and the dashed dot line is the small angle parabolic model.

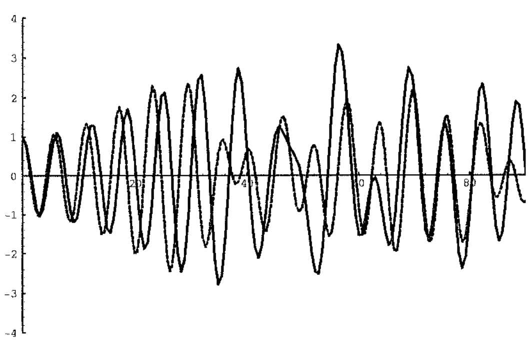


Figure 8: Comparison of the Water Surface Variation Along Outer Wall Between the Exact Solution (solid line) and the Fourier-Galerkin Model (dashed line)



As an example, Figures 4, 4, 2 show the only three wave modes that comprise the total solution shown in Figure 5. The first mode ($n = 0$) has no zero crossing across the channel and the wave action is confined to the outer channel wall. This is the equivalent "whispering gallery" mode. The next mode ($n = 1$) has one zero crossing and a longer angular wave length (defined as $2\pi/\gamma_2$ radians). The last mode has two zero crossings across the channel and an even longer angular wave length. For this case, the channel radii are $r_1 = 75$ m, $r_2 = 100$ m. The channel depth is 4 m and the wave period is 4 s. The corresponding wavenumber $k$ is 0.301 m$^-1$, and therefore the dimensionless channel width is $kw = 7.53$, or 1.2 wave lengths fit across the channel. It is neither a narrow nor a very wide channel. The three values of $\gamma_n$ are 27.6676, 22.8667, 14.2745 from the gravest to the highest mode.

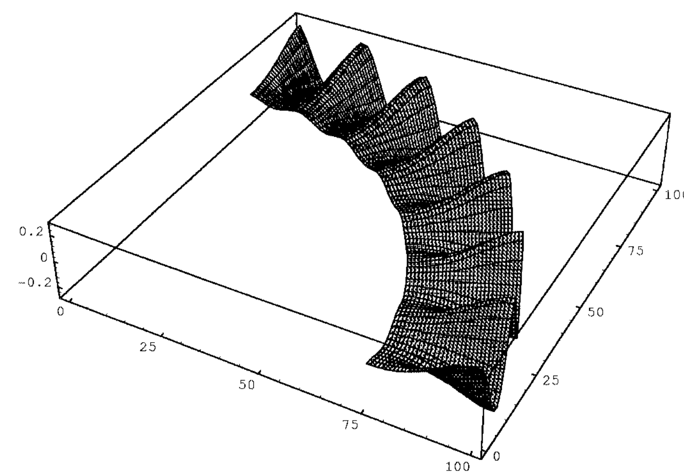


Figure 2: Zeroth (Whispering Gallery) Mode for Analytical Solution

One interesting phenomena of wide channels is the 'loss' of waves within the channel. Since the length of the outer circumference is greater than the inner, there are more waves around the outer circumference than the inner, which means that waves become short-crested across the channel and singular points in the wave phase occur, similar to that discussed by Nye and Berry (1974), Radder (1992), and Dalrymple and Martin (1994).

This problem of progressive waves entering the channel with a given wavenumber $k$ is different than the seiching problem in an annular channel where an integer number of waves fit along the centerline of the circular channel. The

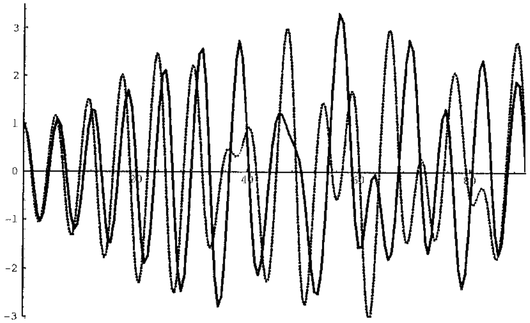

Figure 9: Comparison of the Water Surface Variation Along Outer Wall Between the Exact Solution (solid line) and the Chebyshev-tau Model (dashed line)

Department of Commerce, under Award No. NA 16 RG 0162 (University of Delaware). The U.S. Government is authorized to produce and distribute reprints for government purposes notwithstanding any copyright notation that may appear herein.

## References


Booker, H.G. and Clemmow, P.C., "The Concept of an Angular Spectrum of Plane Waves, and its Relation to that of Polar Diagram and Aperture Distribution," *Proc. Inst. Electr. Engineering,* 3, 97, 11, 1950.

Campbell, I.J., "Wave Motion in an Annular Tank," *Phil. Magazine*, Series 7, 44, 355, 845-853.

Dalrymple, R.A. and J.T. Kirby, "Angular Spectrum Modelling of Water Waves," *Reviews in Aquatic Sciences*, CRC Press, 6, Iss. 5 and 6, 383-404, 1992.

Dalrymple, R.A., K.D. Suh, J.T. Kirby, and J.W. Chae, "Models for Very Wide Angle Water Waves and Wave Diffraction. Part 2. Irregular Bathymetry," *J. Fluid Mechanics*, 201, 299-322, 1989.

Dalrymple, R.A., J.T. Kirby and P.A. Martin, "Spectral Methods for Forward Propagating Water Waves in Conformally-mapped Channels," *Applied Ocean Research*, in press.

Kirby, J.T., R.A. Dalrymple, and H. Kaku, "Parabolic Approximations for Water Waves in Conformal Coordinate Systems,"





*Coastal Engineering,* 23, 185-213, 1994.

Nye, J. F. and M. V. Berry, "Dislocations in wave trains," *Proc. Roy. Soc. London,* A, 336, 165–190, 1974.

Martin, P.A. and R.A. Dalrymple"On Amphidromic Points," *Proc. Roy. Soc. London, A*, 444, 91–104, 1994.

Panchang, V.G. and D. A. Kopriva, "Solution of two-dimensional water-wave propagation problems by Chebyshev collocation," *Mathl. Comput. Modelling,* 12, 625–640, 1989.

Radder, A. C., "Efficient elliptic solvers for the mild-slope equation using the multigrid technique, by B. Li and K. Anastasiou: comments" *Coastal Engineering,* **18**, 347–350, 1992.

Rostafinski, W., "On Propagation of Long Waves in Curved Ducts," *J. Acoustical Soc. of America,*" 52, 5, Pt. 2, 1411-1420, 1972.

Rostafinski, W., "Analysis of Propagation of Waves of Acoustic Frequencies in Curved Ducts," *J. Acoustical Soc. of America,* 56, 1, 11-15, 1974.

Rostafinski, W., "Transmission of Energy in Curved Ducts," *J. Acoustical Soc. of America,*" 56, 3, 1005-1007, 1974.

Rostafinski, W., "Acoustic Systems Containing Curved Duct Sections," *J. Acoustical Soc. of America,*" 60, 1, 23-28, 1976.