Introduction

Our goal is to develop a model capable of predicting the low-frequency wave climate in the surf zone given the offshore wave climate and bottom bathymetry. Presently, we have a fairly good idea of how various low frequency motions (edge waves, shear waves, etc.) are generated in the surf zone (e.g., Bowen and Holman, 1989; Lippmann *et al.*, 1997). However, to be able to predict the low frequency wave climate, we need to know not only how the various low frequency motions are generated but also how they interact with each other.

Based on the linear dispersion relationships for edge and shear waves we can easily identify triads of edge and shear waves that satisfy the resonant interaction conditions

$$\omega_3 = \omega_1 + \omega_2 \qquad \lambda_3 = \lambda_1 + \lambda_2$$

where the ω's represent the frequencies and the λ's represent the wavenumbers. This indicates that significant energy transfer will take place between the various low frequency motions due to triad interactions.

As an example of the importance of the triad interactions, consider the triad of two edge waves and a shear wave sketched in Figure 1. This figure shows that the resonant interaction conditions are satisfied for the triad considered. The existence of this resonance implies the possibility of significant energy transfer to the shear wave from edge wave modes, providing a previously unrecognized mechanism for the generation of shear waves. This mechanism may be capable of explaining observations of shear waves on planar beaches at Santa Barbara and Torrey Pines where stability calculations indicate that the longshore current is stable (Dodd *et al.*, 1992; see also Figure 2).

Both observational (Oltman-Shay *et al.*, 1989) and theoretical (Iranzo *et al.*, 1997) studies suggest that shear waves tend to equilibrate at fairly small amplitudes on nearly planar beaches. Thus, a weakly-nonlinear analysis is a reasonable approach to pursue (on nearly planar beaches). Furthermore, as pointed out earlier, the system supports resonances at second order. The triad interactions which occur at second order are expected to be much stronger effect than the third order self-interaction of shear waves considered by Feddersen (1996). Therefore, we are developing a model that is aimed at accurately describing the energy transfer between the various low frequency motions due to triad interactions. This presentation discusses our initial attack on this problem.

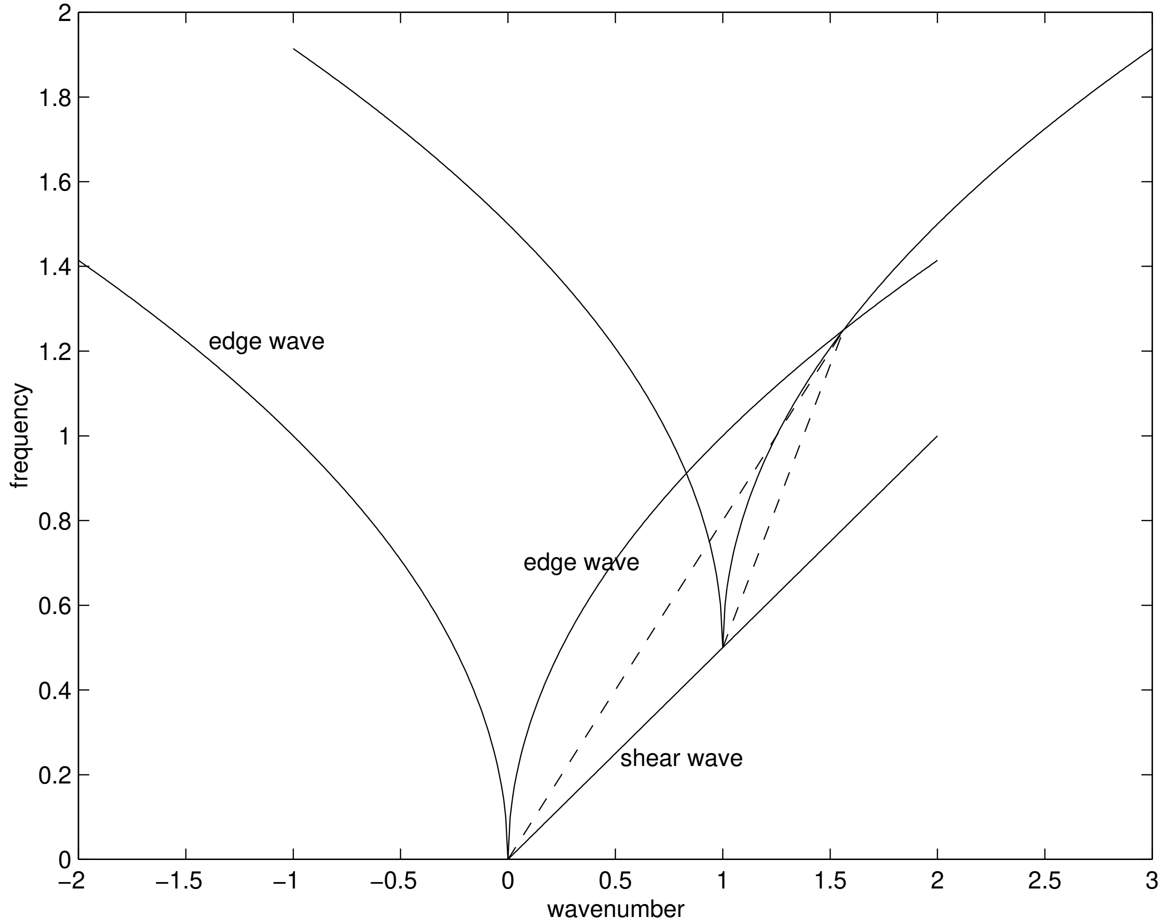

Figure 1: Schematic illustration of resonant interaction between two edge waves and one shear wave. Identifying the shear wave as the first wave in the triad, the origin of the edge-wave dispersion curve is translated up the shear-wave dispersion curve to the locus of wavenumber-frequency for the shear wave being considered. Then, resonances are indicated by intersections between original and new edge-wave dispersion curves. The two dashed lines indicate two edge waves with the same mode number and propagating downstream, which are in resonance with the shear wave.

Model Formulation

The full treatment of the low-frequency wave climatology problem requires a treatment of a wide range of physical phenomena, including the basic edge wave and shear wave motions as well as their coupling to incident and radiated wave fields. In addition, frictional and lateral mixing effects are significant. In our initial attack on the problem, we are considering the simpler situation in which the edge waves are undamped and unforced by external forcing mechanisms. Coupling between the incident wave field and the edge wave climate has recently been described recently by Lippmann *et al.* (1997) and will be added later.

As usual, we assume that the total flow consists of a steady longshore current $V(x)$ and low-frequency waves with surface elevation $\eta(x, y, t)$ and depth-averaged horizontal velocities $u(x, y, t)$ and $v(x, y, t)$ in the cross-shore (x) and longshore (y) directions. We further assume that u and v are much smaller than V. Under these assumptions, the equations governing the free low-frequency waves are (neglecting bottom friction, lateral mixing terms, and radiation stress forcing resulting from averaging over wind-wave time scales)

$$\begin{aligned}
\frac{D\eta}{Dt} + \frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} &= N_c \\
\frac{D(hu)}{Dt} + gh\frac{\partial\eta}{\partial x} &= N_x \\
\frac{D(hv)}{Dt} + hu\frac{dV}{dx} + gh\frac{\partial\eta}{\partial y} &= N_y
\end{aligned}$$

where

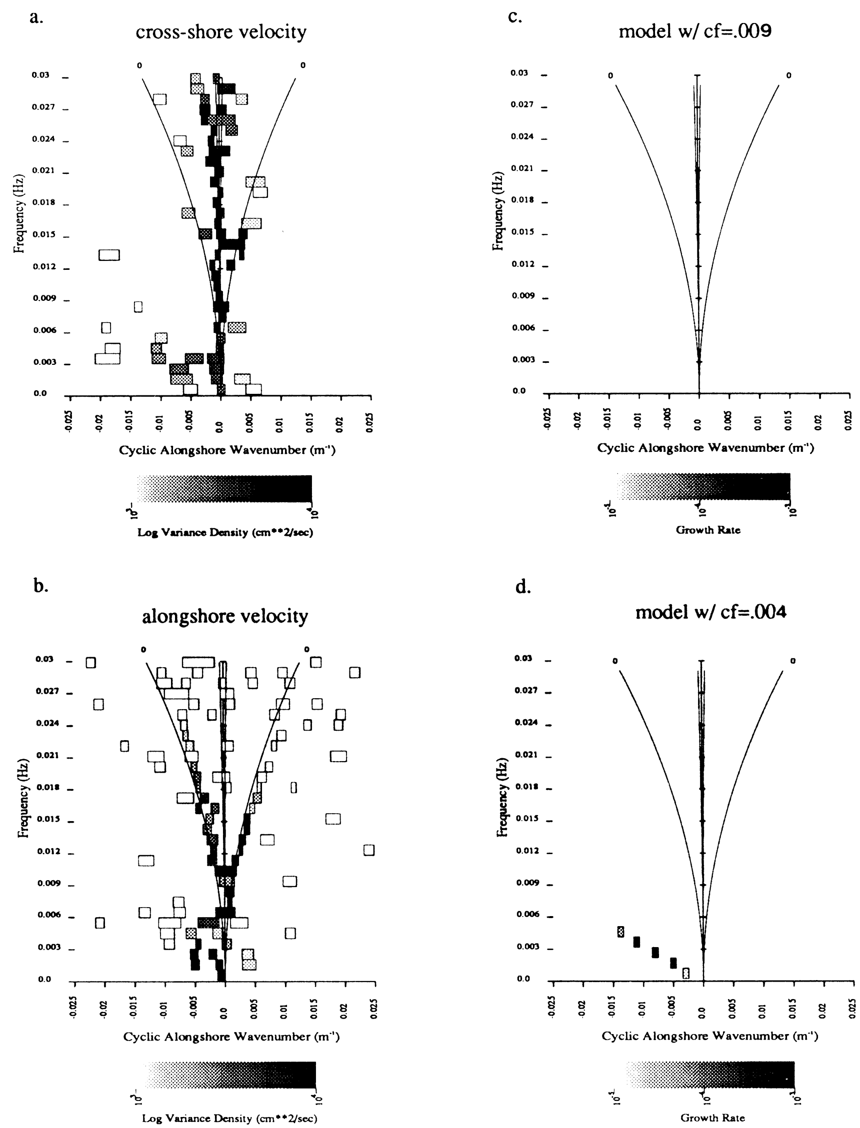

Figure 2: Presence of shear waves on a monotonic beach profile. Longshore current is linearly stable for realistic bottom friction values. (from Dodd *et al.*, 1992)

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V\frac{\partial}{\partial y}$$

The nonlinear terms on the governing equations are given by

$$\begin{aligned}
N_c &= -\left[\frac{\partial(\eta u)}{\partial x} + \frac{\partial(\eta v)}{\partial y}\right] \\
N_x &= -h\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] \\
N_y &= -h\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right]
\end{aligned}$$

To derive equations governing the weakly nonlinear dynamics of low frequency waves, it is convenient to first combine the governing equations to get the following equation in terms of the surface elevation

$$\frac{D}{Dt}\left\{\frac{D^2\eta}{Dt^2} - \frac{\partial}{\partial x}\left(gh\frac{\partial\eta}{\partial x}\right) - gh\frac{\partial^2\eta}{\partial y^2}\right\} + 2gh\frac{dV}{dx}\frac{\partial^2\eta}{\partial x\partial y} = N$$

where N contains the nonlinear terms and is given by

$$N = 2\frac{dV}{dx}\frac{\partial N_y}{\partial y} + \frac{D}{Dt}\left(\frac{DN_c}{Dt} - \frac{\partial N_y}{\partial y} - \frac{\partial N_x}{\partial x}\right)$$

The two basic tasks needed to construct the model are (1) the formulation and solution of the eigenvalue problem for the eigenmodes of the linear problem, and (2) the formulation of the three-wave interaction equations at second order. A related non-trivial problem is the development of an algorithm to search for the resonant interactions in the numerically determined dispersion curves.

Linear Problem

Solutions of the linearized governing equation for surface displacement η are sought in the form

$$\eta(x, y, t) = F(x)\mathrm{e}^{i(\lambda y - \omega t)}$$

where we seek to determine the wave frequency ω given a fixed longshore wavenumber λ. The self-adjoint form of the resulting eigenvalue problem is given by

$$\left(1 - \frac{g\lambda^2 h}{\sigma^2}\right) F + \left(\frac{ghF'}{\sigma^2}\right)' = 0$$

$$F(0) \text{ bounded, } F(x) \to 0 \text{ as } x \to \infty$$

where the relative frequency $\sigma(x)$ is given by

$$\sigma(x) = \omega - \lambda V(x)$$

This formulation was given by Howd *et al.* (1992) but solved there only for edge wave modes. The problem of completely enumerating solutions of this problem is complex, since the system supports both the distortions to the no-current edge waves as well as waves trapped in the longshore current jet and the family of neutrally stable or unstable shear waves. However, orthogonality of the family of solutions $(\omega_n, F_n(x))$ for a fixed wavenumber λ is easily established and is given by

$$\int_0^\infty gh\frac{\sigma_n + \sigma_m}{\sigma_n^2\sigma_m^2}\left[F_n' F_m' + \lambda^2 F_n F_m\right] dx = 0$$

for distinct ω_n, ω_m. The dispersion relation for the nth mode is given by

$$\int_0^\infty \left[1 - \frac{g\lambda^2 h}{\sigma_n^2}\right] F_n^2 dx - \int_0^\infty \frac{gh(F_n')^2}{\sigma_n^2} dx = 0$$

This result reduces to the dispersion relation for edge waves with no longshore current

$$\frac{\omega_n^2}{g} = \frac{\int_0^\infty h(\lambda^2 F_n^2 + (F_n')^2)}{\int_0^\infty F_n^2 dx}$$

given by Pierce and Knobloch (1994). The expression for group velocity is also obtained and is given by

$$C_{gn} = \frac{\int_0^\infty \frac{2gh}{\sigma_n^2}\left[\left(\lambda + \frac{\sigma_n V}{gh}\right) F_n^2 - \left(\frac{V}{\sigma_n}\right)' F_n F_n'\right] dx}{\int_0^\infty \frac{2}{\sigma_n}\left[F_n^2 + \frac{gh\sigma_n'}{\sigma_n^3} F_n F_n'\right] dx}$$

This result also reduces to the expression for edge waves with no longshore current

$$C_{gn} = g\left(\frac{\lambda}{\omega_n}\right) \frac{\int_0^\infty hF_n^2 dx}{\int_0^\infty F_n^2 dx}$$

given by Pierce and Knobloch (1994).

Nonlinear Problem

The results of the previous section may be utilized to describe the leading-order spatial structure and propagation characteristics of each linear component in a random sea of low-frequency waves. The total surface elevation η is expressed as

$$\eta = \sum_{n=1}^{\infty} \epsilon^n \eta_n$$

where $\epsilon \ll 1$. We then expand the η_n's in terms of the eigenfunctions of the linear problem allowing for slow temporal and spatial evolution. Thus, we let

$$\eta_n = \frac{1}{2} \sum_l \sum_p {}^{(n)}A_l^p(T, Y) F_l^p(x) E_l^p(y, t) + *$$

where $*$ indicates a complex conjugate, l is the the index of the dense longshore wavenumber space, p denotes the family of discrete modes at each wavenumber, and $T = \epsilon t$ and $Y = \epsilon y$ represent the slow time and longshore coordinates. The summation above is to be interpreted as a summation over all values of the wave number and all allowable modes. The phase information is contained in the factor

$$E_l^p(y, t) = \exp[i(\lambda_l y - \omega_l^p t)]$$

Substitution of the expansion into the governing equations leads to the linear problem discussed in the earlier section at lowest order $[O(\epsilon)]$. The nature of the linear dispersion relation indicates that the second-order problem is resonantly forced. Using the orthogonality of the modes to eliminate the secular terms leads to coupled evolution equations for the modal amplitudes of the type

$$\frac{\partial^{(1)} A_l^p}{\partial T} + c_{g,l}^p \frac{\partial^{(1)} A_l^p}{\partial Y} = -\frac{2}{I_l^p} \int_0^\infty \frac{\overline{NE_l^{p*}} F_l^p}{(\sigma_l^p)^3} dx$$

where N is the nonlinear term, $c_{g,l}^p$ is the group velocity and I_l^p is an integral displayed earlier as the denominator of the expression for the group velocity. The expression $\overline{NE_l^{p*}}$ has non-trivial contributions of the form

$$\begin{aligned}
\overline{NE_l^{p*}} &= -\frac{i}{8} \sum_m \sum_n \sum_q \sum_r T_{m,n}^{q,r} A_m^q A_n^r \overline{E_m^q E_n^r E_l^{p*}} \\
&= -\frac{i}{8} \sum_m \sum_n \sum_q \sum_r T_{m,-n}^{q,-r} A_m^q A_n^{r*} \overline{E_m^q E_n^{r*} E_l^{p*}} \\
&= -\frac{i}{8} \sum_m \sum_n \sum_q \sum_r T_{-m,n}^{-q,r} A_m^{q*} A_n^r \overline{E_m^{q*} E_n^r E_l^{p*}}
\end{aligned}$$

where contributions arise from triads satisfying any of the resonance conditions

$$\begin{aligned}
\pm\lambda_m \pm \lambda_n - \lambda_l &= 0 \\
Re\{\pm\omega_m^q \pm \omega_n^r - \omega_l^p\} &= 0
\end{aligned}$$

and where the $T_{m,n}^{q,r}$ are complicated interaction coefficients. The first of these conditions allows us to eliminate index n in favor of combinations of l and m. However, the discreteness of the frequency spectrum at each wavenumber still requires an involved algorithm to search for solutions of the resonance condition for frequencies. This algorithm is still under development.

Example Calculations

As an example of the application of the coupled-mode equations, we display the results for the interaction of a single triad of edge waves. The model coefficients are evaluated using the shallow water theory for edge waves on a plane beach with no longshore current, where the mode structure is described by Laguerre polynomials. The system considered consists of (1) a mode 0 edge wave traveling to the right with frequency and wavenumber (ω_1, λ_1), (2) a mode 0 edge wave traveling to the left with frequency and wavenumber $(\omega_1/2, -\lambda_1/4)$, and (3) a mode 1 edge wave travelling to the right with frequency and wavenumber $(\frac{3}{2}\omega_1, \frac{3}{4}\lambda_1)$. In this system, the mode 1 wave represents the high-frequency member of the triad.

In terms of surface displacement amplitude, a, at the shoreline, the set of three coupled equations may be written as

$$\begin{aligned}
a_{1,T} &= \frac{i}{8}\frac{\omega_1^3}{g\beta^2}a_2^* a_3 \\
a_{2,T} &= \frac{i}{64}\frac{\omega_1^3}{g\beta^2}a_1^* a_3 \\
a_{3,T} &= \frac{9i}{64}\frac{\omega_1^3}{g\beta^2}a_1^* a_3
\end{aligned}$$

where T denotes slow time and β is the planar beach slope. The system of equations has a simple energy integral,

$$E = |a_1|^2 + |a_2|^2 + |a_3|^2 = \text{constant}$$

In the first picture, the system is initialized with $a_1 = 1, a_2 = 1, a_3 = 0$. There is a rapid exchange of energy between the co-directional mode 0 (a_1) and mode 1 (a_3) waves, with energy moving readily from low frequency to high frequency and back. The mode 0 wave travelling in the opposite direction (a_2) facilitates the interaction but plays little role in the energy exchange.

In the next picture, the system is initialized with $a_1 = .1, a_2 = .1, a_3 = 1$, and most energy is initially situated in the higher mode wave. The energy exchange is slower in this case but still leads to nearly complete exchange of energy between the lowest and highest frequency waves, and is still largely confined to interaction between the co-directional waves.

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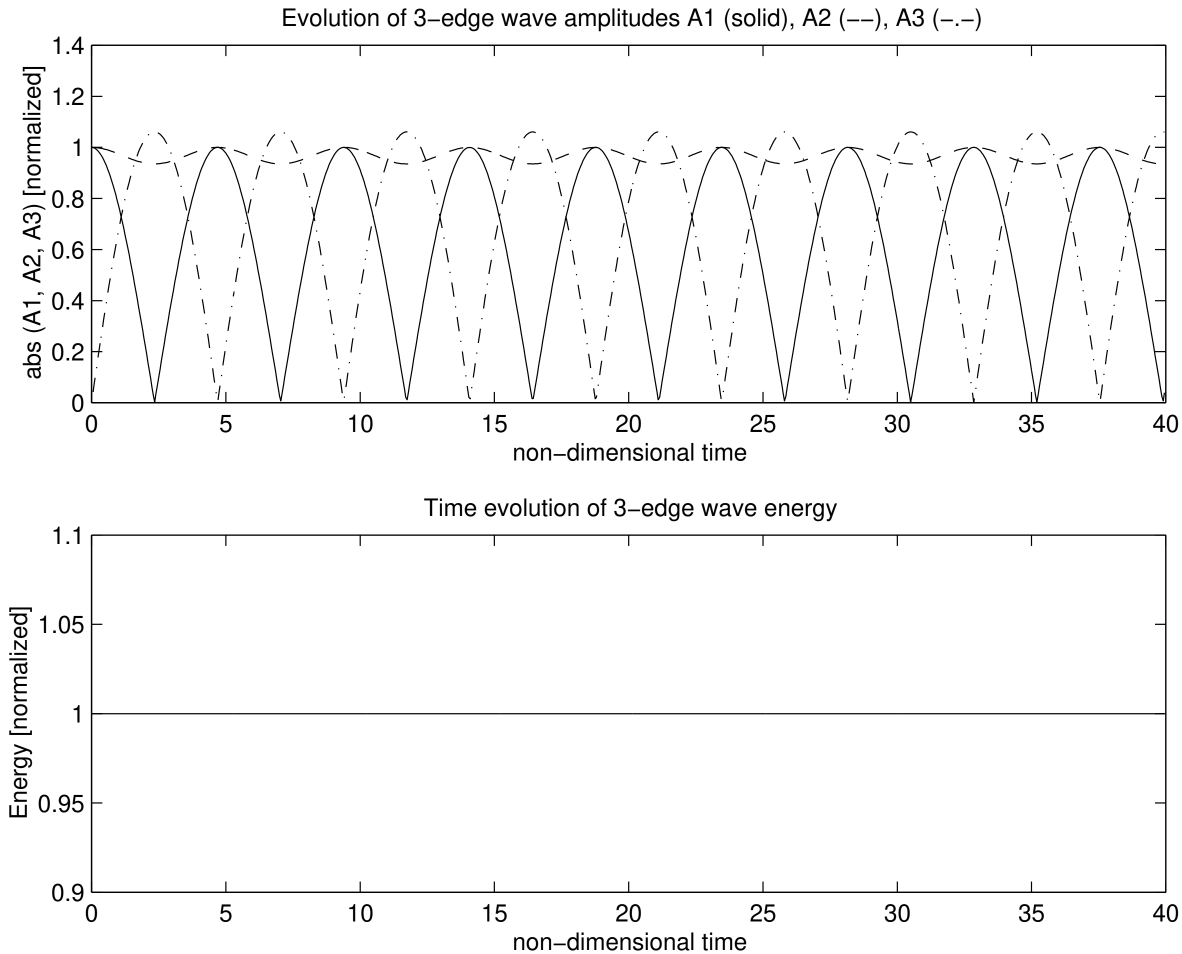

Figure 3: Energy exchange between three resonantly interacting edge waves. Energy initially concentrated at low frequency, low mode number.

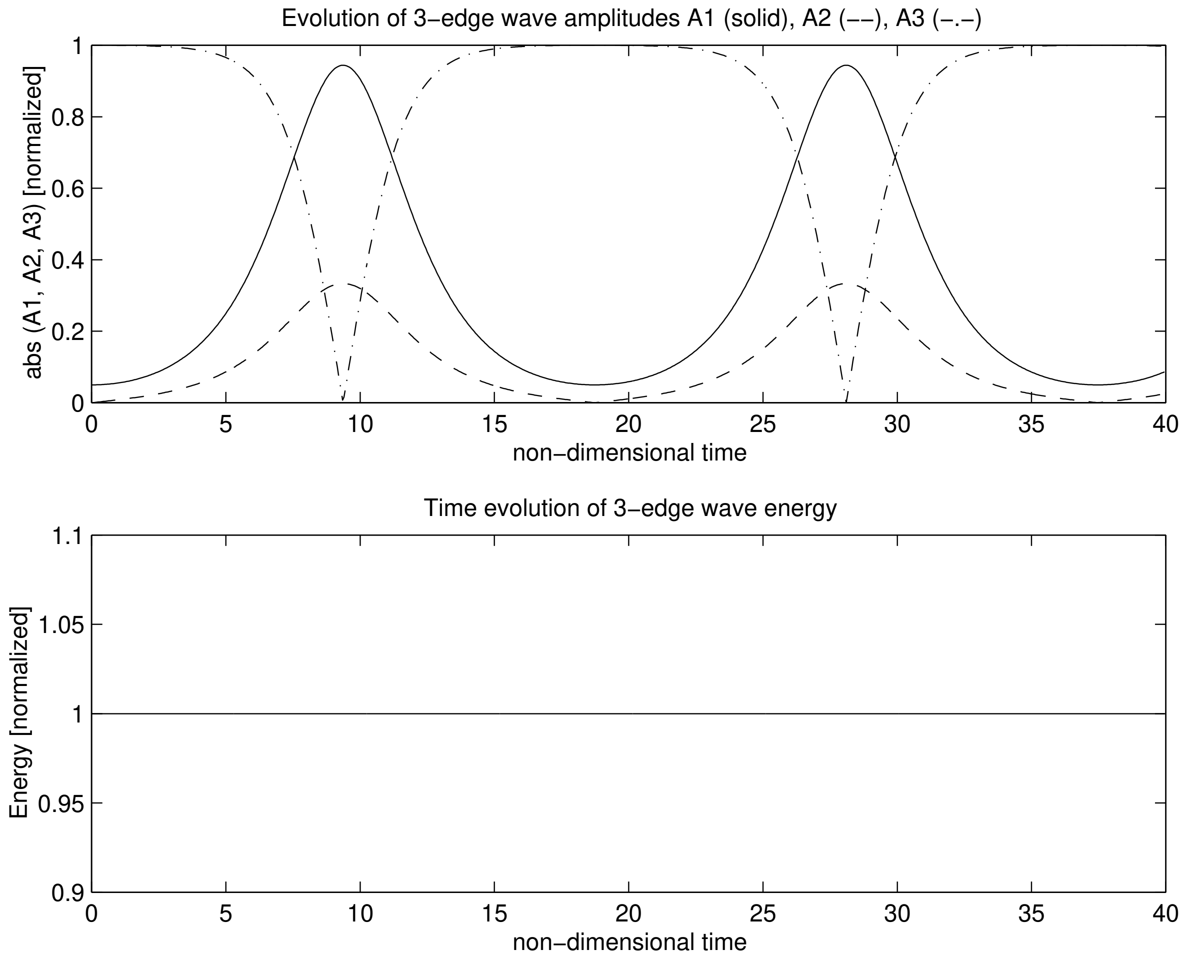

Figure 4: Energy exchange between three resonantly interacting edge waves. Energy initially concentrated at high frequency, high mode number.